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ELEMENTS

OF

TRIGONOMETRY

WITH TABLES

BY

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CENTRAL MANUAL TRAINING SCHOOL  
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PHILADELPHIA

D. ANSON PARTRIDGE

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## PREFACE.

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THE introduction and first five chapters of this book have been prepared for the use of beginners in the subject. The second appendix has been added for those intending to take up work in higher departments of Mathematics.

The subject of logarithms is usually considered in treatises on algebra; but for the convenience of those not familiar with the theory of this subject, the first appendix has been added.

To aid in a clear understanding of each process in the various investigations, the aim has been to closely associate with every equation a definite meaning with reference to a diagram.

The tables used are in general as precise as the data usually obtained by students in engineering, physics, or chemistry. It is assumed that a few seven-place tables are accessible to a class; and it is recommended that some classroom work be done with seven-place tables, especially in the solution of spherical triangles.

It is hoped that the answers are free from error. The numerical computations were made with care by the author, but the results have not been verified by any one else.

Any corrections or suggestions relating to the book will be thankfully received.

H. C. WHITAKER.

PHILADELPHIA, May, 1898.



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# A TABLE

OF THE

## NATURAL VALUES OF THE TRIGONOMETRIC FUNCTIONS OF ANGLES

FROM  $0^{\circ}$  TO  $90^{\circ}$  FOR EVERY  $10'$

ALSO

# A TABLE

## FOR CONVERTING THE SEXAGESIMAL MEASURE OF AN ANGLE INTO CIRCULAR MEASURE

ALSO

# A TABLE

## FOR CONVERTING CIRCULAR FUNCTIONS INTO HYPERBOLIC FUNCTIONS

## NATURAL VALUES OF FUNCTIONS.

| $x$       | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$  | $\cot x$  | $\cos x$ |            |
|-----------|----------|----------|----------|-----------|-----------|----------|------------|
| $0^\circ$ | .0 00000 | .0 00000 | 1.00 00  | Infinite. | Infinite. | 1.00 00  | $90^\circ$ |
| $10'$     | 02909    | 02909    | 00       | 343.775   | 343.774   | 00       | $50'$      |
| $20'$     | 05818    | 05818    | 00       | 171.888   | 171.885   | 00       | $40'$      |
| $30'$     | 08727    | 08727    | 00       | 114.593   | 114.589   | 00       | $30'$      |
| $40'$     | 11635    | 11636    | 01       | 85.946    | 85.940    | .99 99   | $20'$      |
| $50'$     | 14544    | 14545    | 01       | 68.757    | 68.750    | 99       | $10'$      |
| $1^\circ$ | .0 1745  | .0 1746  | 1.00 02  | 57.299    | 57.290    | .99 98   | $89^\circ$ |
| $10'$     | 2036     | 2036     | 02       | 49.114    | 49.104    | 98       | $50'$      |
| $20'$     | 2327     | 2328     | 03       | 42.976    | 42.964    | 97       | $40'$      |
| $30'$     | 2618     | 2619     | 03       | 38.202    | 38.188    | 97       | $30'$      |
| $40'$     | 2908     | 2910     | 04       | 34.382    | 34.368    | 96       | $20'$      |
| $50'$     | 3199     | 3201     | 05       | 31.258    | 31.242    | 95       | $10'$      |
| $2^\circ$ | .0 3490  | .0 3492  | 1.00 06  | 28.654    | 28.636    | .99 94   | $88^\circ$ |
| $10'$     | 3781     | 3783     | 07       | 26.451    | 26.432    | 93       | $50'$      |
| $20'$     | 4071     | 4075     | 08       | 24.562    | 24.542    | 92       | $40'$      |
| $30'$     | 4362     | 4366     | 10       | 22.926    | 22.904    | 90       | $30'$      |
| $40'$     | 4653     | 4658     | 11       | 21.494    | 21.470    | 89       | $20'$      |
| $50'$     | 4943     | 4949     | 12       | 20.230    | 20.206    | 88       | $10'$      |
| $3^\circ$ | .0 5234  | .0 5241  | 1.00 14  | 19.107    | 19.081    | .99 86   | $87^\circ$ |
| $10'$     | 5524     | 5533     | 15       | 18.103    | 18.075    | 85       | $50'$      |
| $20'$     | 5814     | 5824     | 17       | 17.198    | 17.169    | 83       | $40'$      |
| $30'$     | 6105     | 6116     | 19       | 16.380    | 16.350    | 81       | $30'$      |
| $40'$     | 6395     | 6408     | 21       | 15.637    | 15.605    | 80       | $20'$      |
| $50'$     | 6685     | 6700     | 22       | 14.958    | 14.924    | 78       | $10'$      |
| $4^\circ$ | .0 6976  | .0 6993  | 1.00 24  | 14.336    | 14.301    | .99 76   | $86^\circ$ |
| $10'$     | 7266     | 7285     | 26       | 13.763    | 13.727    | 74       | $50'$      |
| $20'$     | 7556     | 7578     | 29       | 13.235    | 13.197    | 71       | $40'$      |
| $30'$     | 7846     | 7870     | 31       | 12.745    | 12.706    | 69       | $30'$      |
| $40'$     | 8136     | 8163     | 33       | 12.291    | 12.250    | 67       | $20'$      |
| $50'$     | 8426     | 8456     | 36       | 11.868    | 11.826    | 64       | $10'$      |
| $5^\circ$ | .0 8716  | .0 8749  | 1.00 38  | 11.474    | 11.430    | .99 62   | $85^\circ$ |
| $10'$     | 9005     | 9042     | 41       | 11.105    | 11.059    | 59       | $50'$      |
| $20'$     | 9295     | 9335     | 43       | 10.758    | 10.712    | 57       | $40'$      |
| $30'$     | 9585     | 9629     | 46       | 10.433    | 10.385    | 54       | $30'$      |
| $40'$     | 9874     | 9923     | 49       | 10.128    | 10.078    | 51       | $20'$      |
| $50'$     | .1 0164  | .1 0216  | 52       | 9.8391    | 9.7882    | 48       | $10'$      |
| $6^\circ$ | .1 0453  | .1 0510  | 1.00 55  | 9.5668    | 9.5144    | .99 45   | $84^\circ$ |
| $10'$     | 0742     | 0805     | 58       | 9.3092    | 9.2553    | 42       | $50'$      |
| $20'$     | 1031     | 1099     | 61       | 9.0652    | 9.0098    | 39       | $40'$      |
| $30'$     | 1320     | 1394     | 65       | 8.8337    | 8.7769    | 36       | $30'$      |
| $40'$     | 1609     | 1688     | 68       | 8.6138    | 8.5555    | 32       | $20'$      |
| $50'$     | 1898     | 1983     | 72       | 8.4047    | 8.3450    | 29       | $10'$      |
| $7^\circ$ | .1 2187  | .1 2278  | 1.00 75  | 8.2055    | 8.1443    | .99 25   | $83^\circ$ |
|           | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$  | $\tan y$  | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>7°</b>  | .1 219   | .1 228   | 1.00 75  | 8. 2055  | 8. 1443  | .99 25   | <b>83°</b> |
| 10'        | 248      | 257      | 79       | 0156     | 7. 9530  | 22       | 50'        |
| 20'        | 276      | 287      | 82       | 7. 8344  | 7704     | 18       | 40'        |
| 30'        | 305      | 317      | 86       | 6613     | 5958     | 14       | 30'        |
| 40'        | 334      | 346      | 90       | 4957     | 4287     | 11       | 20'        |
| 50'        | 363      | 376      | 94       | 3372     | 2687     | 07       | 10'        |
| <b>8°</b>  | .1 392   | .1 405   | 1.00 98  | 7. 1853  | 7. 1154  | .99 03   | <b>82°</b> |
| 10'        | 421      | 435      | 1.01 02  | 0396     | 6. 9682  | .98 99   | 50'        |
| 20'        | 449      | 465      | 07       | 6. 8998  | 8269     | 94       | 40'        |
| 30'        | 478      | 495      | 11       | 7655     | 6912     | 90       | 30'        |
| 40'        | 507      | 524      | 15       | 6363     | 5606     | 86       | 20'        |
| 50'        | 536      | 554      | 20       | 5121     | 4348     | 81       | 10'        |
| <b>9°</b>  | .1 564   | .1 584   | 1.01 25  | 6. 3925  | 6. 3138  | .98 77   | <b>81°</b> |
| 10'        | 593      | 614      | 29       | 2772     | 1970     | 72       | 50'        |
| 20'        | 622      | 644      | 34       | 1661     | 0844     | 68       | 40'        |
| 30'        | 650      | 673      | 39       | 0589     | 5. 9758  | 63       | 30'        |
| 40'        | 679      | 703      | 44       | 5. 9554  | 8708     | 58       | 20'        |
| 50'        | 708      | 733      | 49       | 8554     | 7694     | 53       | 10'        |
| <b>10°</b> | .1 736   | .1 763   | 1.01 54  | 5. 7588  | 5. 6713  | .98 48   | <b>80°</b> |
| 10'        | 765      | 793      | 60       | 6653     | 5764     | 43       | 50'        |
| 20'        | 794      | 823      | 65       | 5749     | 4845     | 38       | 40'        |
| 30'        | 822      | 853      | 70       | 4874     | 3955     | 33       | 30'        |
| 40'        | 851      | 883      | 76       | 4026     | 3093     | 27       | 20'        |
| 50'        | 880      | 914      | 81       | 3205     | 2257     | 22       | 10'        |
| <b>11°</b> | .1 908   | .1 944   | 1.01 87  | 5. 2408  | 5. 1446  | .98 16   | <b>79°</b> |
| 10'        | 937      | 974      | 93       | 1636     | 0658     | 11       | 50'        |
| 20'        | 965      | .2 004   | 99       | 0886     | 4. 9894  | 05       | 40'        |
| 30'        | 994      | 035      | 1.02 05  | 0159     | 9152     | .97 99   | 30'        |
| 40'        | .2 022   | 065      | 11       | 4. 9452  | 8430     | 93       | 20'        |
| 50'        | 051      | 095      | 17       | 8765     | 7729     | 87       | 10'        |
| <b>12°</b> | .2 079   | .2 126   | 1.02 23  | 4. 8097  | 4. 7046  | .97 81   | <b>78°</b> |
| 10'        | 108      | 156      | 30       | 7448     | 6382     | 75       | 50'        |
| 20'        | 136      | 186      | 36       | 6817     | 5736     | 69       | 40'        |
| 30'        | 164      | 217      | 43       | 6202     | 5107     | 63       | 30'        |
| 40'        | 193      | 247      | 49       | 5604     | 4494     | 57       | 20'        |
| 50'        | 221      | 278      | 56       | 5022     | 3897     | 50       | 10'        |
| <b>13°</b> | .2 250   | .2 309   | 1.02 63  | 4. 4454  | 4. 3315  | .97 44   | <b>77°</b> |
| 10'        | 278      | 339      | 70       | 3901     | 2747     | 37       | 50'        |
| 20'        | 306      | 370      | 77       | 3362     | 2193     | 30       | 40'        |
| 30'        | 334      | 401      | 84       | 2837     | 1653     | 24       | 30'        |
| 40'        | 363      | 432      | 91       | 2324     | 1126     | 17       | 20'        |
| 50'        | 391      | 462      | 99       | 1824     | 0611     | 10       | 10'        |
| <b>14°</b> | .2 419   | .2 493   | 1.03 06  | 4. 1336  | 4. 0108  | .97 03   | <b>76°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>14°</b> | .2 419   | .2 493   | 1.03 06  | 4. 1336  | 4. 0108  | .97 03   | <b>76°</b> |
| 10'        | 447      | 524      | 14       | 0859     | 3. 9617  | .96 96   | 50'        |
| 20'        | 476      | 555      | 21       | 0394     | 9136     | 89       | 40'        |
| 30'        | 504      | 586      | 29       | 3. 9939  | 8667     | 81       | 30'        |
| 40'        | 532      | 617      | 37       | 9495     | 8208     | 74       | 20'        |
| 50'        | 560      | 648      | 45       | 9061     | 7760     | 67       | 10'        |
| <b>15°</b> | .2 588   | .2 679   | 1.03 53  | 3. 8637  | 3. 7321  | .96 59   | <b>75°</b> |
| 10'        | 616      | 711      | 61       | 8222     | 6891     | 52       | 50'        |
| 20'        | 644      | 742      | 69       | 7817     | 6470     | 44       | 40'        |
| 30'        | 672      | 773      | 77       | 7420     | 6059     | 36       | 30'        |
| 40'        | 700      | 805      | 86       | 7032     | 5656     | 28       | 20'        |
| 50'        | 728      | 836      | 94       | 6652     | 5261     | 21       | 10'        |
| <b>16°</b> | .2 756   | .2 867   | 1.04 03  | 3. 6280  | 3. 4874  | .96 13   | <b>74°</b> |
| 10'        | 784      | 899      | 12       | 5915     | 4495     | 05       | 50'        |
| 20'        | 812      | 931      | 21       | 5559     | 4124     | .95 96   | 40'        |
| 30'        | 840      | 962      | 29       | 5209     | 3759     | 88       | 30'        |
| 40'        | 868      | 994      | 39       | 4867     | 3402     | 80       | 20'        |
| 50'        | 896      | .3 026   | 48       | 4532     | 3052     | 72       | 10'        |
| <b>17°</b> | .2 924   | .3 057   | 1.04 57  | 3. 4203  | 3. 2709  | .95 63   | <b>73°</b> |
| 10'        | 952      | 089      | 66       | 3881     | 2371     | 55       | 50'        |
| 20'        | 979      | 121      | 76       | 3565     | 2041     | 46       | 40'        |
| 30'        | .3 007   | 153      | 85       | 3255     | 1716     | 37       | 30'        |
| 40'        | 035      | 185      | 95       | 2951     | 1397     | 28       | 20'        |
| 50'        | 062      | 217      | 1.05 05  | 2653     | 1084     | 20       | 10'        |
| <b>18°</b> | .3 090   | .3 249   | 1.05 15  | 3. 2361  | 3. 0777  | .95 11   | <b>72°</b> |
| 10'        | 118      | 281      | 25       | 2074     | 0475     | 02       | 50'        |
| 20'        | 145      | 314      | 35       | 1792     | 0178     | .94 92   | 40'        |
| 30'        | 173      | 346      | 45       | 1515     | 2. 9887  | 83       | 30'        |
| 40'        | 201      | 378      | 55       | 1244     | 9600     | 74       | 20'        |
| 50'        | 228      | 411      | 66       | 0977     | 9319     | 65       | 10'        |
| <b>19°</b> | .3 256   | .3 443   | 1.05 76  | 3. 0716  | 2. 9042  | .94 55   | <b>71°</b> |
| 10'        | 283      | 476      | 87       | 0458     | 8770     | 46       | 50'        |
| 20'        | 311      | 508      | 98       | 0206     | 8502     | 36       | 40'        |
| 30'        | 338      | 541      | 1.06 08  | 2. 9957  | 8239     | 26       | 30'        |
| 40'        | 365      | 574      | 19       | 9713     | 7980     | 17       | 20'        |
| 50'        | 393      | 607      | 31       | 9474     | 7725     | 07       | 10'        |
| <b>20°</b> | .3 420   | .3 640   | 1.06 42  | 2. 9238  | 2. 7475  | .93 97   | <b>70°</b> |
| 10'        | 448      | 673      | 53       | 9006     | 7228     | 87       | 50'        |
| 20'        | 475      | 706      | 65       | 8779     | 6985     | 77       | 40'        |
| 30'        | 502      | 739      | 76       | 8555     | 6746     | 67       | 30'        |
| 40'        | 529      | 772      | 88       | 8334     | 6511     | 56       | 20'        |
| 50'        | 557      | 805      | 1.07 00  | 8117     | 6279     | 46       | 10'        |
| <b>21°</b> | .3 584   | .3 839   | 1.07 11  | 2. 7904  | 2. 6051  | .93 36   | <b>69°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>21°</b> | .3 584   | .3 839   | 1.07 11  | 2. 7904  | 2. 6051  | .93 36   | <b>69°</b> |
| 10'        | 611      | 872      | 23       | 7695     | 5826     | 25       | 50'        |
| 20'        | 638      | 906      | 36       | 7488     | 5605     | 15       | 40'        |
| 30'        | 665      | 939      | 48       | 7285     | 5386     | 04       | 30'        |
| 40'        | 692      | 973      | 60       | 7085     | 5172     | .92 93   | 20'        |
| 50'        | 719      | .4 006   | 73       | 6888     | 4960     | 83       | 10'        |
| <b>22°</b> | .3 746   | .4 040   | 1.07 85  | 2. 6695  | 2. 4751  | .92 72   | <b>68°</b> |
| 10'        | 773      | 074      | 98       | 6504     | 4545     | 61       | 50'        |
| 20'        | 800      | 108      | 1.08 11  | 6316     | 4342     | 50       | 40'        |
| 30'        | 827      | 142      | 24       | 6131     | 4142     | 39       | 30'        |
| 40'        | 854      | 176      | 37       | 5949     | 3945     | 28       | 20'        |
| 50'        | 881      | 210      | 50       | 5770     | 3750     | 16       | 10'        |
| <b>23°</b> | .3 907   | .4 245   | 1.08 64  | 2. 5593  | 2. 3559  | .92 05   | <b>67°</b> |
| 10'        | 934      | 279      | 77       | 5419     | 3369     | .91 94   | 50'        |
| 20'        | 961      | 314      | 91       | 5247     | 3183     | 82       | 40'        |
| 30'        | 987      | 348      | 1.09 04  | 5078     | 2998     | 71       | 30'        |
| 40'        | .4 014   | 383      | 18       | 4912     | 2817     | 59       | 20'        |
| 50'        | 041      | 417      | 32       | 4748     | 2637     | 47       | 10'        |
| <b>24°</b> | .4 067   | .4 452   | 1.09 46  | 2. 4586  | 2. 2460  | .91 35   | <b>66°</b> |
| 10'        | 094      | 487      | 61       | 4426     | 2286     | 24       | 50'        |
| 20'        | 120      | 522      | 75       | 4269     | 2113     | 12       | 40'        |
| 30'        | 147      | 557      | 89       | 4114     | 1943     | 00       | 30'        |
| 40'        | 173      | 592      | 1.10 04  | 3961     | 1775     | .90 88   | 20'        |
| 50'        | 200      | 628      | 19       | 3811     | 1609     | 75       | 10'        |
| <b>25°</b> | .4 226   | .4 663   | 1.10 34  | 2. 3662  | 2. 1445  | .90 63   | <b>65°</b> |
| 10'        | 253      | 699      | 49       | 3515     | 1283     | 51       | 50'        |
| 20'        | 279      | 734      | 64       | 3371     | 1123     | 38       | 40'        |
| 30'        | 305      | 770      | 79       | 3228     | 0965     | 26       | 30'        |
| 40'        | 331      | 806      | 95       | 3087     | 0809     | 13       | 20'        |
| 50'        | 358      | 841      | 1.11 10  | 2949     | 0655     | 01       | 10'        |
| <b>26°</b> | .4 384   | .4 877   | 1.11 26  | 2. 2812  | 2. 0503  | .89 88   | <b>64°</b> |
| 10'        | 410      | 913      | 42       | 2677     | 0353     | 75       | 50'        |
| 20'        | 436      | 950      | 58       | 2543     | 0204     | 62       | 40'        |
| 30'        | 462      | 986      | 74       | 2412     | 0057     | 49       | 30'        |
| 40'        | 488      | .5 022   | 90       | 2282     | 1. 9912  | 36       | 20'        |
| 50'        | 514      | 059      | 1.12 07  | 2153     | 9768     | 23       | 10'        |
| <b>27°</b> | .4 540   | .5 095   | 1.12 23  | 2. 2027  | 1. 9626  | .89 10   | <b>63°</b> |
| 10'        | 566      | 132      | 40       | 1902     | 9486     | .88 97   | 50'        |
| 20'        | 592      | 169      | 57       | 1779     | 9347     | 84       | 40'        |
| 30'        | 617      | 206      | 74       | 1657     | 9210     | 70       | 30'        |
| 40'        | 643      | 243      | 91       | 1537     | 9074     | 57       | 20'        |
| 50'        | 669      | 280      | 1.13 08  | 1418     | 8940     | 43       | 10'        |
| <b>28°</b> | .4 695   | .5 317   | 1.13 26  | 2. 1301  | 1. 8807  | .88 29   | <b>62°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>28°</b> | .4 695   | .5 317   | 1.1 326  | 2.1 301  | 1.8 807  | .88 29   | <b>62°</b> |
| 10'        | 720      | 354      | 343      | 185      | 676      | 16       | 50'        |
| 20'        | 746      | 392      | 361      | 070      | 546      | 02       | 40'        |
| 30'        | 772      | 430      | 379      | 2.0 957  | 418      | .87 88   | 30'        |
| 40'        | 797      | 467      | 397      | 846      | 291      | 74       | 20'        |
| 50'        | 823      | 505      | 415      | 736      | 165      | 60       | 10'        |
| <b>29°</b> | .4 848   | .5 543   | 1.1 434  | 2.0 627  | 1.8 040  | .87 46   | <b>61°</b> |
| 10'        | 874      | 581      | 452      | 519      | 1.7 917  | 32       | 50'        |
| 20'        | 899      | 619      | 471      | 413      | 796      | 18       | 40'        |
| 30'        | 924      | 658      | 490      | 308      | 675      | 04       | 30'        |
| 40'        | 950      | 696      | 509      | 204      | 556      | .86 89   | 20'        |
| 50'        | 975      | 735      | 528      | 101      | 437      | 75       | 10'        |
| <b>30°</b> | .5 000   | .5 774   | 1.1 547  | 2.0 000  | 1.7 321  | .86 60   | <b>60°</b> |
| 10'        | 025      | 812      | 566      | 1.9 900  | 205      | 46       | 50'        |
| 20'        | 050      | 851      | 586      | 801      | 090      | 31       | 40'        |
| 30'        | 075      | 890      | 606      | 703      | 1.6 977  | 16       | 30'        |
| 40'        | 100      | 930      | 626      | 606      | 864      | 01       | 20'        |
| 50'        | 125      | 969      | 646      | 511      | 753      | .85 87   | 10'        |
| <b>31°</b> | .5 150   | .6 009   | 1.1 666  | 1.9 416  | 1.6 643  | .85 72   | <b>59°</b> |
| 10'        | 175      | 048      | 687      | 323      | 534      | 57       | 50'        |
| 20'        | 200      | 088      | 707      | 230      | 426      | 42       | 40'        |
| 30'        | 225      | 128      | 728      | 139      | 319      | 26       | 30'        |
| 40'        | 250      | 168      | 749      | 048      | 212      | 11       | 20'        |
| 50'        | 275      | 208      | 770      | 1.8 959  | 107      | .84 96   | 10'        |
| <b>32°</b> | .5 299   | .6 249   | 1.1 792  | 1.8 871  | 1.6 003  | .84 80   | <b>58°</b> |
| 10'        | 324      | 289      | 813      | 783      | 1.5 900  | 65       | 50'        |
| 20'        | 348      | 330      | 835      | 697      | 798      | 50       | 40'        |
| 30'        | 373      | 371      | 857      | 612      | 697      | 34       | 30'        |
| 40'        | 398      | 412      | 879      | 527      | 597      | 18       | 20'        |
| 50'        | 422      | 453      | 901      | 443      | 497      | 03       | 10'        |
| <b>33°</b> | .5 446   | .6 494   | 1.1 924  | 1.8 361  | 1.5 399  | .83 87   | <b>57°</b> |
| 10'        | 471      | 536      | 946      | 279      | 301      | 71       | 50'        |
| 20'        | 495      | 577      | 969      | 198      | 204      | 55       | 40'        |
| 30'        | 519      | 619      | 992      | 118      | 108      | 39       | 30'        |
| 40'        | 544      | 661      | 1.2 015  | 039      | 013      | 23       | 20'        |
| 50'        | 568      | 703      | 039      | 1.7 960  | 1.4 919  | 07       | 10'        |
| <b>34°</b> | .5 592   | .6 745   | 1.2 062  | 1.7 883  | 1.4 826  | .82 90   | <b>56°</b> |
| 10'        | 616      | 787      | 086      | 806      | 733      | 74       | 50'        |
| 20'        | 640      | 830      | 110      | 730      | 641      | 58       | 40'        |
| 30'        | 664      | 873      | 134      | 655      | 550      | 41       | 30'        |
| 40'        | 688      | 916      | 158      | 581      | 460      | 25       | 20'        |
| 50'        | 712      | 959      | 183      | 507      | 370      | 08       | 10'        |
| <b>35°</b> | .5 736   | .7 002   | 1.2 208  | 1.7 434  | 1.4 281  | .81 92   | <b>55°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>35°</b> | .5 736   | .7 002   | 1.2 208  | 1.7 434  | 1.4 281  | .81 92   | <b>55°</b> |
| 10'        | 760      | 046      | 233      | 362      | 193      | 75       | 50'        |
| 20'        | 783      | 089      | 258      | 291      | 106      | 58       | 40'        |
| 30'        | 807      | 133      | 283      | 221      | 019      | 41       | 30'        |
| 40'        | 831      | 177      | 309      | 151      | 1.3 934  | 24       | 20'        |
| 50'        | 854      | 221      | 335      | 081      | 848      | 07       | 10'        |
| <b>36°</b> | .5 878   | .7 265   | 1.2 361  | 1.7 013  | 1.3 764  | .80 90   | <b>54°</b> |
| 10'        | 901      | 310      | 387      | 1.6 945  | 680      | 73       | 50'        |
| 20'        | 925      | 355      | 413      | 878      | 597      | 56       | 40'        |
| 30'        | 948      | 400      | 440      | 812      | 514      | 39       | 30'        |
| 40'        | 972      | 445      | 467      | 746      | 432      | 21       | 20'        |
| 50'        | 995      | 490      | 494      | 681      | 351      | 04       | 10'        |
| <b>37°</b> | .6 018   | .7 536   | 1.2 521  | 1.6 616  | 1.3 270  | .79 86   | <b>53°</b> |
| 10'        | 041      | 581      | 549      | 553      | 190      | 69       | 50'        |
| 20'        | 065      | 627      | 577      | 489      | 111      | 51       | 40'        |
| 30'        | 088      | 673      | 605      | 427      | 032      | 34       | 30'        |
| 40'        | 111      | 720      | 633      | 365      | 1.2 954  | 16       | 20'        |
| 50'        | 134      | 766      | 661      | 303      | 876      | .78 98   | 10'        |
| <b>38°</b> | .6 157   | .7 813   | 1.2 690  | 1.6 243  | 1.2 799  | .78 80   | <b>52°</b> |
| 10'        | 180      | 860      | 719      | 183      | 723      | 62       | 50'        |
| 20'        | 202      | 907      | 748      | 123      | 647      | 44       | 40'        |
| 30'        | 225      | 954      | 778      | 064      | 572      | 26       | 30'        |
| 40'        | 248      | .8 002   | 807      | 005      | 497      | 08       | 20'        |
| 50'        | 271      | 050      | 837      | 1.5 948  | 423      | .77 90   | 10'        |
| <b>39°</b> | .6 293   | .8 098   | 1.2 868  | 1.5 890  | 1.2 349  | .77 71   | <b>51°</b> |
| 10'        | 316      | 146      | 898      | 833      | 276      | 53       | 50'        |
| 20'        | 338      | 195      | 929      | 777      | 203      | 35       | 40'        |
| 30'        | 361      | 243      | 960      | 721      | 131      | 16       | 30'        |
| 40'        | 383      | 292      | 991      | 666      | 059      | .76 98   | 20'        |
| 50'        | 406      | 342      | 1.3 022  | 611      | 1.1 988  | 79       | 10'        |
| <b>40°</b> | .6 428   | .8 391   | 1.3 054  | 1.5 557  | 1.1 918  | .76 60   | <b>50°</b> |
| 10'        | 450      | 441      | 086      | 504      | 847      | 42       | 50'        |
| 20'        | 472      | 491      | 118      | 450      | 778      | 23       | 40'        |
| 30'        | 494      | 541      | 151      | 398      | 708      | 04       | 30'        |
| 40'        | 517      | 591      | 184      | 345      | 640      | .75 85   | 20'        |
| 50'        | 539      | 642      | 217      | 294      | 571      | 66       | 10'        |
| <b>41°</b> | .6 561   | .8 693   | 1.3 250  | 1.5 243  | 1.1 504  | .75 47   | <b>49°</b> |
| 10'        | 583      | 744      | 284      | 192      | 436      | 28       | 50'        |
| 20'        | 604      | 796      | 318      | 141      | 369      | 09       | 40'        |
| 30'        | 626      | 847      | 352      | 092      | 303      | .74 90   | 30'        |
| 40'        | 648      | 899      | 386      | 042      | 237      | 70       | 20'        |
| 50'        | 670      | 952      | 421      | 1.4 993  | 171      | 51       | 10'        |
| <b>42°</b> | .6 691   | .9 004   | 1.3 456  | 1.4 945  | 1.1 106  | .74 31   | <b>48°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

| $x$        | $\sin x$ | $\tan x$ | $\sec x$ | $\csc x$ | $\cot x$ | $\cos x$ |            |
|------------|----------|----------|----------|----------|----------|----------|------------|
| <b>42°</b> | .6 691   | .9 004   | 1.3 456  | 1.4 945  | 1.1 106  | .74 31   | <b>48°</b> |
| 10'        | 713      | 057      | 492      | 897      | 041      | 12       | 50'        |
| 20'        | 734      | 110      | 527      | 849      | 1.0 977  | .73 92   | 40'        |
| 30'        | 756      | 163      | 563      | 802      | 913      | 73       | 30'        |
| 40'        | 777      | 217      | 600      | 755      | 850      | 53       | 20'        |
| 50'        | 799      | 271      | 636      | 709      | 786      | 33       | 10'        |
| <b>43°</b> | .6 820   | .9 325   | 1.3 673  | 1.4 663  | 1.0 724  | .73 14   | <b>47°</b> |
| 10'        | 841      | 380      | 711      | 617      | 661      | .72 94   | 50'        |
| 20'        | 862      | 435      | 748      | 572      | 599      | 74       | 40'        |
| 30'        | 884      | 490      | 786      | 527      | 538      | 54       | 30'        |
| 40'        | 905      | 545      | 824      | 483      | 477      | 34       | 20'        |
| 50'        | 926      | 601      | 863      | 439      | 416      | 14       | 10'        |
| <b>44°</b> | .6 947   | .9 657   | 1.3 902  | 1.4 396  | 1.0 355  | .71 93   | <b>46°</b> |
| 10'        | 967      | 713      | 941      | 352      | 295      | 73       | 50'        |
| 20'        | 988      | 770      | 980      | 310      | 235      | 53       | 40'        |
| 30'        | .7 009   | 827      | 1.4 020  | 267      | 176      | 33       | 30'        |
| 40'        | 030      | 884      | 061      | 225      | 117      | 12       | 20'        |
| 50'        | 050      | 942      | 101      | 183      | 058      | .70 92   | 10'        |
| <b>45°</b> | .7 071   | 1.0 000  | 1.4 142  | 1.4 142  | 1.0 000  | .70 71   | <b>45°</b> |
|            | $\cos y$ | $\cot y$ | $\csc y$ | $\sec y$ | $\tan y$ | $\sin y$ | $y$        |

CIRCULAR {  $\theta = \frac{x}{180^\circ} \pi$ .

MEASURE.

HYPERBOLIC {  $\sinh u = \tan x$ .

FUNCTIONS. {  $\cosh u = \sec x$ .

| $x$  | $\theta$ | $x$  | $\theta$ | $x$ | $u$   | $x$ | $u$    |
|------|----------|------|----------|-----|-------|-----|--------|
| 1°   | .01745   | 1'   | .00029   | 2°  | .0349 | 40° | .7629  |
| 2°   | .03491   | 2'   | .00058   | 4°  | .0699 | 42° | .8092  |
| 3°   | .05236   | 3'   | .00087   | 6°  | .1049 | 44° | .8569  |
| 4°   | .06981   | 4'   | .00116   | 8°  | .1401 | 46° | .9063  |
| 5°   | .08727   | 5'   | .00145   | 10° | .1754 | 48° | .9575  |
| 6°   | .10472   | 6'   | .00175   | 12° | .2110 | 50° | 1.0107 |
| 7°   | .12217   | 7'   | .00204   | 14° | .2468 | 52° | 1.0662 |
| 8°   | .13963   | 8'   | .00233   | 16° | .2830 | 54° | 1.1242 |
| 9°   | .15708   | 9'   | .00262   | 18° | .3195 | 56° | 1.1851 |
| 10°  | .17453   | 10'  | .00291   | 20° | .3564 | 58° | 1.2492 |
| 20°  | .34907   | 20'  | .00582   | 22° | .3938 | 60° | 1.3170 |
| 30°  | .52360   | 30'  | .00873   | 24° | .4317 | 62° | 1.3890 |
| 40°  | .69813   | 40'  | .01164   | 26° | .4702 | 64° | 1.4659 |
| 50°  | .87266   | 50'  | .01454   | 28° | .5094 | 66° | 1.5485 |
| 60°  | 1.04720  | 60'  | .01745   | 30° | .5493 | 68° | 1.6379 |
| 70°  | 1.22173  |      |          | 32° | .5900 | 70° | 1.7354 |
| 80°  | 1.39626  | 10'' | .00005   | 34° | .6317 | 72° | 1.8427 |
| 90°  | 1.57080  | 20'' | .00010   | 36° | .6743 | 74° | 1.9623 |
| 100° | 1.74533  | 40'' | .00019   | 38° | .7180 | 76° | 2.0973 |

# ELEMENTS OF TRIGONOMETRY.

## INTRODUCTION.

Trigonometry investigates the numerical relations between segments,\* angles, and areas of figures.

**1. The meaning of geometric operations** will be retained in trigonometry. Thus, Fig. 1, if the segment  $CD$ , of length  $a$ , is added to the segment  $DE$ , of length  $b$ , the **sum** is the segment  $CE$ , of length  $a + b$ .



FIG. 1.

In Fig. 2, if  $DE$ , of length  $b$ , is subtracted from  $CD$ , of length  $a$ , the **remainder**  $CE$  is of length  $a - b$ .

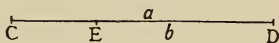


FIG. 2.

In Fig. 3, if  $CD$  is of length  $a$ ,  $CE = a + a + a + a = 4a$ .



FIG. 3.

In Fig. 4, if  $CD$  is of length  $a$ ,  $CE$  (two of the ten equal parts of  $CD$ ) is of length  $.2a$ .

Hence if a segment of length  $a$  is multiplied by 2.4, the

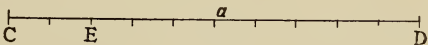


FIG. 4.

**product** will be considered either  $a$  divided by 10 and the

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\* In this work a straight line, or simply a line, is conceived to be infinitely long in two opposite directions from a point; hence a half-line is conceived to be infinitely long in one direction from a point; and a segment of a line, or simply a segment, is the name which will be applied to a limited portion of a line.

quotient multiplied by 24, or  $a$  multiplied by 24 and the product divided by 10.

In Fig. 5,  $EF$ , of length  $b$ , may be conceived as being divided into  $CD$ , of length  $a$ . If  $CD$  contains  $EF$  five times, the ratio of  $CD$  to  $EF$  is said to be 5.

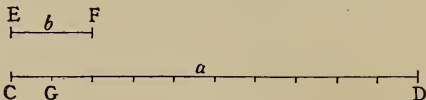


FIG. 5.

If it is desired to obtain the ratio of  $EF$  to  $CD$ ,  $CD$  is first divided into 10 equal parts and one of these parts  $CG$  applied to  $EF$ . If  $EF$  contains  $CG$  exactly two times, the ratio of  $EF$  to  $CD$  is said to be .2. The ratios  $\frac{a}{b} = 5$  and  $\frac{b}{a} = .2$  have a product of 1, and are said to be reciprocal ratios.

In Fig. 6,  $EF$  is not contained exactly in  $CD$ , but is contained twice and a remainder  $GD$  more than three-tenths of  $EF$ , but less than three and one-half tenths; the ratio of  $CD$  to  $EF$  is said to be 2.3+.

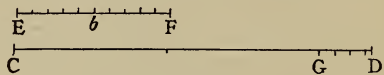


FIG. 6.

This value of the ratio contains an error not greater than .05. The ratio of the error to the assumed value is not greater than .05 in 2.3, or 5 in 230, or 1 in 46. If this error in 2.3, the assumed value of the ratio, is too great for use in a problem in which the ratio enters,  $EF$  is divided into 100, 1000, 10,000, etc., equal parts, and one of them applied to  $GD$ . Thus, if  $EF$  is divided into 10,000 parts and  $GD$  is found to contain one of these parts not less than 3470.5 times, but less than 3471 times, the ratio of  $CD$  to  $EF$  is said to be 2.3471-. This value of the ratio contains an error not greater than .00005. The ratio of the error to the assumed value is not greater than .00005 in 2.3471, or 1 in 46,942.

In the same way, if the ratio of  $CD$  to  $EF$  is desired,  $CD$

is divided into 10, 100, 1000, 10,000, etc., equal parts, and so divided applied to  $EF$ . Suppose  $CD$  is divided into 10,000 parts, and the beginning of  $CD$  is placed at  $E$ . Then if the 4261st division of  $CD$  is the one nearest to  $F$ , the ratio of  $EF$  to  $CD$  is said to be approximately .4261; the ratio of the error to the assumed value of the ratio in this case is not greater than 1 in 8522.

**Operations upon angles.** — All that has been said with reference to addition, subtraction, multiplication, and division of segments will apply to corresponding operations performed upon angles. Thus, Fig. 7, the sum of angle  $BAC(=a)$  and angle  $CAD(=b)$  is the angle  $BAD(=a+b)$ .

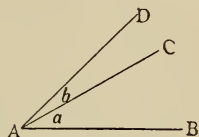


FIG. 7.

Similarly, the difference of  $BAC(=a)$  and  $CAD(=b)$ , Fig. 8, is the angle  $BAD(=a-b)$ .

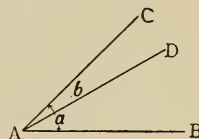


FIG. 8.

In a similar manner, the interpretation of the multiplication of an angle by a number, the division of an angle by a number, and the division of an angle by another angle will be conceived to be the same as in geometry.

## NEGATIVE SEGMENTS AND ANGLES.

**2.** Assume the direction of measurement of the segment  $CD$ , of length  $a$ , Fig. 9, to be from left to right. From  $D$ , the terminal point of  $CD$ , lay off a segment  $DE$ , of length  $b$ , backward on  $CD$ . Then if  $b$  is greater than  $a$ ,  $E$  will be to the left of  $C$ , but  $a-b$  will still be said to be the length of the segment  $CE$ , measured from  $C$  to  $E$ , or in the direction opposite to that assumed to be positive. Now the length of  $a$  in the discussion is

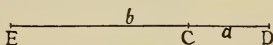


FIG. 9.

arbitrary, and may be conceived to be equal to the length of  $CD$  in Fig. 10, or zero; in which case the segment  $CE$  (measured from right to left, or opposite the direction assumed to be positive) is equal to  $-b$ .

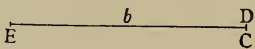


FIG. 10.

Hence, if any segment is **multiplied by  $-1$** , the effect of the operation is to measure the segment from the same point as at first but in an opposite direction; thus,  $-1 \times a = -a$ , and  $-1 \times -a = +a$ .

The multiplication of a segment by a positive number will be considered to be repetition of the segment in the same direction the number of times indicated by the multiplier (which may be either integral or fractional); thus, 4.3 times  $-a = -(4.3 a)$ . The product of a segment by a negative number will be conceived to be: first, a reversal of the direction of the segment, and then the multiplication by a positive number; thus,

$$-4.3 \times a = 4.3 \times -1 \times a = 4.3 \times -a = -(4.3 a).$$

Also

$$-4.3 \times -a = 4.3 \times -1 \times -a = 4.3 \times a = +(4.3 a).$$

If division is defined as being the process of obtaining a quantity, which, multiplied with the divisor, will produce the dividend, each process of multiplication gives

$$\frac{-(4.3 a)}{-a} = 4.3.$$

$$\frac{-(4.3 a)}{4.3} = -a.$$

$$\frac{-(4.3 a)}{a} = -4.3.$$

$$\frac{-(4.3 a)}{-4.3} = a.$$

$$\frac{+(4.3 a)}{-a} = -4.3.$$

$$\frac{+(4.3 a)}{-4.3} = -a.$$

The same methods of reasoning will give the conception of **negative angles**. Thus, if turning a screw in one direction is positive turning, then turning the screw in the opposite direction is negative turning, because it cancels or undoes

an equal amount of positive turning; and negative angles so conceived may be multiplied by positive or negative numbers, or divided by positive or negative numbers or angles.

### PRODUCTS AND QUOTIENTS OF SEGMENTS.

**3.** In geometry, the product of two segments is explained to be a term used to aid us in counting the number of units of area in a surface; thus if a rectangle has sides of length 3 inches and 2 inches, respectively, the area is said to be 3 inches  $\times$  2 inches = 6 square inches. This statement is a short way of saying that in the rectangle there can be arranged 2 rows of inch-squares, 3 in a row, making a total of 6 inch-squares or 6 square inches. In the same way the operation 4 inches  $\times$  3 inches  $\times$  2 inches = 24 cubic inches, is a method of counting cubes in a rectangular parallelopiped. In trigonometry, the same meaning is attached to these expressions as in geometry. Thus, the product of a segment and an abstract number, say  $4.3a$ , is said to be a length; the product (so called) of two segments and an abstract number, say  $4.3a^2$  or  $4.3ab$ , is said to be an area; the product of three segments and an abstract number, say  $4.3abc$  or  $4.3a^2b$ , is said to be a volume.

Since division is the inverse of multiplication, an area is indicated by the quotient of a volume by a segment, such as  $\frac{4.3abc}{5.2d}$ . Similarly,  $\frac{4.3abc}{5.2de}$ ,  $\frac{4.3ab}{5.2c}$ ,  $\sqrt{4.3ab}$ ,  $\sqrt[3]{4.3abc}$  represent segments.

**Expressions having no geometric meaning.** — In geometry, expressions for lengths, areas, and volumes have meaning; but many forms which occur in trigonometric reductions such as  $\sqrt{a}$ ,  $\frac{4.3}{a}$ ,  $\sqrt{abc}$ ,  $4.3abcd$ , are said to have no geometric meaning.

**Dimensions.** — Any expression representing a length is said to be of one dimension; an expression representing an area is said to be of two dimensions; and any expression is of  $k$  dimensions if the algebraic sum of the exponents of the segments in each term is  $k$ . Thus a ratio has zero dimensions, since it is of the form  $\frac{a}{b} = a^1b^{-1}$ ;  $\frac{2}{a}$  has  $-1$  dimensions;  $\sqrt[3]{ab} = a^{\frac{1}{3}}b^{\frac{1}{3}}$  is of  $\frac{2}{3}$  dimensions;  $abcd$  is of 4 dimensions; and so on.

**4. Homogeneity of trigonometric equations.** — No geometric meaning can be attached to the result obtained by adding an area and a length; or by adding a volume and a length; or by adding a volume and an area. And any equation involving terms of different dimensions is also without meaning. Hence in every trigonometric equation, each term must have the same dimensions. Equations in which this condition is fulfilled are called homogeneous equations. Therefore every trigonometric equation must be a homogeneous equation.

This property of trigonometric equations is frequently of value as a check on the correctness of reduction processes. For example, suppose an investigation of the relations of the parts of any figure is undertaken. The first equation used must be homogeneous; any other equation derived from it should also be homogeneous; if it fails in homogeneity a mistake has been made in the process of reduction.

**Exercises.** — 1. What are the dimensions of each of the following expressions:

$$\sqrt[3]{\frac{7}{5} \frac{a^2bc}{de}}, \quad \sqrt{\frac{3abc^2}{8d}}, \quad \frac{a^kb^l}{c^m}, \quad a^{\frac{1}{m}}b^{\frac{1}{n}}c.$$

2. Write an equation containing six terms of different form, each term representing a segment; also an equation containing six terms, each representing an area.

**5. Significant figures.**—Zeros are not significant figures, if they are supplied to a number merely for the purpose of putting the decimal point in the right position. Thus in the number .00908, the two zeros immediately following the decimal point are not significant figures, but the remaining figures, 9, 0, and 8, are significant figures.

Also, in writing approximate values of large numbers, zeros are frequently supplied when necessary to get the first significant figure of the number in the proper position with reference to the unit's place. Thus in the number 92,500,000 miles, the 5 is understood to be the nearest figure in the hundred-thousands place; the zeros following the 5 indicate a lack of knowledge as to the value of the figures in the ten-thousands place and in lower orders; the 9, 2, and 5 are the only significant figures in this number.

**6. Errors in approximate values of numbers.**—Suppose an assumed value of a number, expressed with four significant figures, the fourth figure being the nearest figure in that place. The ratio of the error to the assumed value of the number is not greater than 1 to 2000. For the most unfavorable case is when 1000 is assumed as the value of 1000.5, in which case the ratio is 1 to 2000.

In a similar manner the ratio of the error of an approximate value of a number of five significant figures (the fifth figure being the nearest figure) cannot be greater than 1 to 20,000; and so on.

**7. Errors arising in the processes of calculation.**—Suppose that assumed values of numbers are introduced into a numerical calculation. From the principles of arithmetic all numbers arising subsequently in the process of calculation will in general have about the same degree of precision as was contained in the assumed value of the least degree of precision which entered into the calculation. Thus, consider the following examples:

**Multiplication.**—Suppose it is desired to multiply the approximate value .7862 by the number 43.59. The number of which .7862 is the approximate value is between .78615 and .78625. Multiply each separately by 43.59. The products are 34.2682785 and 34.2726375. It will be seen that 34.27 will represent either product to four significant figures. Now multiply .7862 by 43.59 by first multiplying by 40, then by 3, then by .5, then by .09.

$$\begin{array}{r}
 .7862 \\
 43.59 \\
 \hline
 31.448\,0 \\
 2.358\,6 \\
 .393\,10 \\
 .070\,758 \\
 \hline
 34.270\,458
 \end{array}$$

$$\begin{array}{r}
 .7862 \\
 43.59 \\
 \hline
 31.448\,0 \\
 2\,359 \\
 393 \\
 71 \\
 \hline
 34.27
 \end{array}$$

To reject all figures to the right of the vertical line drawn through the products, use all the figures of the multiplicand when multiplying by 40, and put in the decimal point; in multiplying by 3, strike out the 2 of the multiplicand, but use it to obtain figures which may affect the column to the left of the vertical line; thus, 3 times 2 are 6, and 6 being more than 5, 1 is carried; 3 times 6 are 18, and 1 to carry is 19, and so on. In multiplying by 5, the 6 of the multiplicand is struck out, but is used to obtain 3 to carry; thus, 5 times 6 are 30, 5 times 8 are 40, and 3 to carry are 43, and so on. In multiplying by 9, the 8 is struck out, but the 86 now struck out being nearly 90,

we proceed: 9 times 6 are 54, 9 times 8 are 72, and 5 to carry are 77, 9 times 7 are 63, and 8 to carry are 71; or simply considering the 86 thrown away to be 90, we say 9 times 9 are 81, 9 times 7 are 63, and 8 to carry are 71. In adding the partial products, the right-hand column adds to 21; reject the 1 and carry the 2 to the next column, as only four significant figures of the product can be correctly obtained. The method is expressed in the following

**Rule.**—Use the figures of the multiplier in reverse order. Supply the decimal point in the first partial product. Reject all unnecessary figures.

**Division.**—To reverse the process, divide the product 34.27 by 78.62.

## COMMON METHOD.

.7862)34.27 00000(43.589

$$\begin{array}{r}
 31\ 44\ 8 \\
 \hline
 2\ 82\ 20 \\
 2\ 35\ 86 \\
 \hline
 46\ 340 \\
 39\ 310 \\
 \hline
 7\ 0300 \\
 6\ 2896 \\
 \hline
 74040 \\
 70758
 \end{array}$$

## CONTRACTED METHOD.

.7862)34.27  $\times \times$  (43.59

$$\begin{array}{r}
 31\ 45\ \times \\
 \hline
 2\ 82 \\
 2\ 36 \\
 \hline
 46 \\
 39 \\
 \hline
 7
 \end{array}$$

Crosses are added to make the number of decimal places in the dividend equal to the number in the divisor. After the multiplication by the quotient figure 4, put as many crosses after the 4 as there are unused crosses in the dividend. Then supply the decimal point. The rest of the process is similar to the process of multiplication.

The square root of .7862.

## COMMON METHOD.

.78620000(.88667

$$\begin{array}{r}
 64 \\
 168 \overline{)1462} \\
 \underline{1344} \\
 1766 \overline{)11800} \\
 \underline{10596} \\
 17726 \overline{)120400} \\
 \underline{106356} \\
 17732 \overline{)14044}
 \end{array}$$

## CONTRACTED METHOD.

.7862(.8867

$$\begin{array}{r}
 64 \\
 168 \overline{)1462} \\
 \underline{1344} \\
 176 \overline{)118} \\
 \underline{106} \\
 12
 \end{array}$$

A consideration of these examples will show that if data are correct to only three significant figures (say an error of 1 in 1000), the calculation may be kept in four significant figures; for the result will be as precise as the data, and further figures are unnecessary and unreliable.

Similarly, if the data are correct to four significant figures (say an error of 1 in 10,000), the calculation may be kept in five significant figures. And so on.

**Applications.** — The summation of series is one of the many practical applications of approximate values of numbers. Thus, suppose it is desired to compute to ten decimal places the value of the series :

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \text{etc.},$$

in which each term is obtained from the one preceding by division, the divisors increasing by 1 each time. The work proceeds as in the margin. It will be seen that the nearest tenth decimal figure is 5.

|     |                  |
|-----|------------------|
| 1)  | 1.               |
| 2)  | 1.               |
| 3)  | .5               |
| 4)  | .166,666,666,67  |
| 5)  | 41,666,666,67    |
| 6)  | 8,333,333,33     |
| 7)  | 1,388,888,89     |
| 8)  | 198,412,70       |
| 9)  | 24,801,59        |
| 10) | 2,755,73         |
| 11) | 275,57           |
| 12) | 25,05            |
| 13) | 2,09             |
| 14) | 16               |
|     | 1                |
|     | <hr/>            |
|     | 2.718,281,828,46 |

**Problems.** — Perform the following multiplications, keeping only the figures which affect the fourth significant figure of the product. After each multiplication, perform the reverse operation of division or square root:

$$234.1 \times 4.444 = 1040$$

$$.4567 \times 9.009 = 4.114$$

$$7060 \times .0302 = 213.2$$

$$.7854 \times 34.25 = 26.90$$

$$1.414 \times 1.414 = 2.000$$

$$.7454 \times .7454 = .5000$$

$$57.33 \times 57.33 = 3287$$

$$1.012 \times 1.012 = 1.024$$

## CHAPTER I.

### FUNCTIONS OF ANGLES.

**8. Generation of an angle.** — The angle  $BAC$  may be conceived to be generated by a turning of a half-line from either side of the angle to the other side. Thus, the position  $AC$  may be reached by a turning of a half-line from  $AB$  through an angle  $x +$  any number of revolutions; or the same position may be reached by a turning from  $AB$  in an opposite direction through an angle  $360^\circ - x +$  any number of revolutions. Also, if  $b$ , the length of  $AC$ , is regarded as negative, the turning from  $AB$  is either  $180^\circ + x$  or a downward turning of  $180^\circ - x$ . For example, if, in Fig. 11,  $x$  is  $30^\circ$  and  $b$  is 10 inches, the point  $C$  is determined by any of the following values of  $b$  and  $x$ :

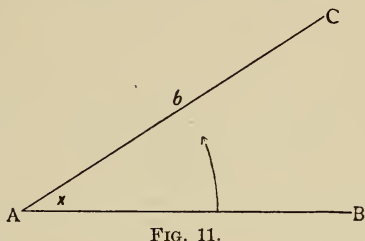
$$b = +10; x = 30^\circ, 390^\circ, 750^\circ, 1110^\circ, \text{ etc.}$$

$$b = -10; x = 210^\circ, 570^\circ, 930^\circ, 1290^\circ, \text{ etc.}$$

$$b = +10; x = -330^\circ, -690^\circ, -1050^\circ, \text{ etc.}$$

$$b = -10; x = -150^\circ, -510^\circ, -870^\circ, \text{ etc.}$$

The half-line  $AB$  from which angles will be conceived to be measured will be called the **initial line**; the final position  $AC$  of the revolving half-line will be called the **terminal**



line of the angle; the length,  $b$ , from the vertex of the angle to any point  $C$  on the terminal line will be called the distance of  $C$ .

The direction of the initial line is entirely arbitrary. It will be drawn to the right in the discussions in this chapter. The angle will be regarded as positive when generated by a half-line turning from the initial line upward to the left, as indicated by the arrow, Fig. 11. It will be regarded as negative when generated by a half-line turning in the opposite direction. The distance will be regarded as positive when laid off from the vertex of the angle along the terminal line.

**9. The four quadrants.** — When a positive angle is under consideration, it will be conceived to be moved so that its vertex is brought at  $A$ , one side coincides with  $AB$ , and the angle extends in a positive direction. Then if the terminal line falls in the angle  $BAD$  (where  $DD_1$  is perpendicular to  $BAB_1$ ), the angle is said to be of the first quadrant; if the terminal line falls in  $DAB_1$ , the angle is of the second quadrant; if the terminal line is in  $B_1AD_1$ , the angle is of the third quadrant; if the terminal line is in  $D_1AB$ , the angle is of the fourth quadrant. Thus,

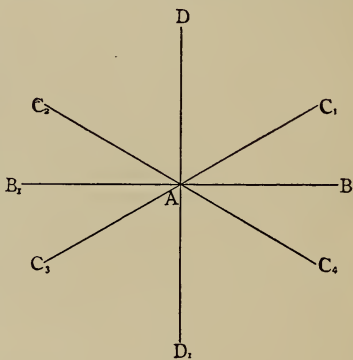


FIG. 12.

$30^\circ, 390^\circ, -330^\circ, -690^\circ$  are angles of the first quadrant.  
 $150^\circ, 510^\circ, -210^\circ, -570^\circ$  are angles of the second quadrant.  
 $210^\circ, 570^\circ, -150^\circ, -510^\circ$  are angles of the third quadrant.  
 $330^\circ, 690^\circ, -30^\circ, -390^\circ$  are angles of the fourth quadrant.

**Abscissa and ordinate.** — From any point  $C$  drop a perpendicular upon the initial line  $AB$ . The distance from  $A$  to the foot of this perpendicular is called the abscissa of  $C$ . The length of the perpendicular is called the ordinate of  $C$ .

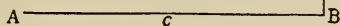


FIG. 13.

Abscissæ measured to the right of  $A$  will be considered positive, and measured to the left, negative.

Ordinates measured upward from the initial line (or initial line drawn backward) will be considered positive, and measured downward, negative.

Thus,

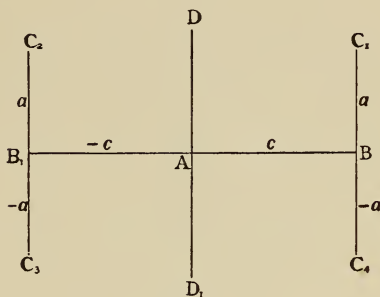


FIG. 14.

In Quadrant I,  $a$  is  $+$  and  $c$  is  $+$

In Quadrant II,  $a$  is  $+$  and  $c$  is  $-$ .

In Quadrant III,  $a$  is  $-$  and  $c$  is  $-$ .

In Quadrant IV,  $a$  is  $-$  and  $c$  is  $+$ .

On the half-line  $AB$ ,  $a$  is zero and  $c$  is  $+$ .

On the half-line  $AD$ ,  $a$  is  $+$  and  $c$  is zero.

On the half-line  $AB_1$ ,  $a$  is zero and  $c$  is  $-$ .

On the half-line  $AD_1$ ,  $a$  is  $-$  and  $c$  is zero.

**Definition of function.** — Suppose a magnitude  $m$ , and a number of other magnitudes,  $p$ ,  $q$ ,  $r$ , so related to  $m$ , that when  $m$  changes in value, all of the others change in value;

and when  $m$  remains fixed in value, all of the others remain fixed. In such a case,  $p, q, r$  are said to be functions of  $m$ .

**10. The six ratios.**—If we take in pairs the ordinate, distance, and abscissa of a point on the terminal line of an angle  $A$ , six different ratios can be obtained :

$$\frac{a}{b}, \frac{a}{c}, \frac{b}{c}, \frac{b}{a}, \frac{c}{a}, \frac{c}{b}$$

It will now be shown that these six ratios are functions of the angle  $A$ .

For suppose, first, that  $b$  remains the same length while the angle  $A$  changes in magnitude; say  $A$  increases. Then if  $A$  is an angle of the first quadrant, the point  $C$  will move upward to the left, its ordinate will increase, and its abscissa will decrease.

Hence  $\frac{a}{b}$ ,  $\frac{a}{c}$ , and  $\frac{b}{c}$  will in-

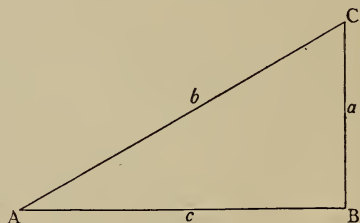


FIG. 15.

crease, and their reciprocal ratios will decrease.

In a similar manner by assuming  $a$  or  $c$  to remain the same length, or by assuming the moving point  $C$  to be in any other quadrant, the same thing may be shown.

Assume now that  $C$  moves so that the angle  $A$  is unchanged. If  $C$  moves either toward  $A$  or away from it to a point  $C'$ , the similar triangles  $ABC$  and  $AB'C'$  give:

$$BC : AC = B'C' : AC'$$

$$BC : AB = B'C' : AB'$$

$$AC : AB = AC' : AB'$$

the reciprocal ratios also being equal in pairs.

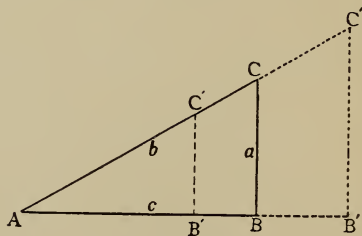


FIG. 16.

**Names of the ratios.** — For any point on the terminal line of an angle,

The ratio of the ordinate to the distance is called the **sine** of the angle. The sine of  $x$  is written  $\sin x$ .

The ratio of the ordinate to the abscissa is called the **tangent** of the angle. The tangent of  $x$  is written  $\tan x$ .

The ratio of the distance to the abscissa is called the **secant** of the angle. The secant of  $x$  is written  $\sec x$ .

These three ratios connect the ordinate, distance, and abscissa of a point; but it is often convenient to use the reciprocal ratios.

Imagine the triangle  $ABC_1$  to be taken up from the plane and placed so that the vertex  $C_1$  falls at  $A$ , and the side  $C_1B$  coincides with the initial line.  $B$  takes the position

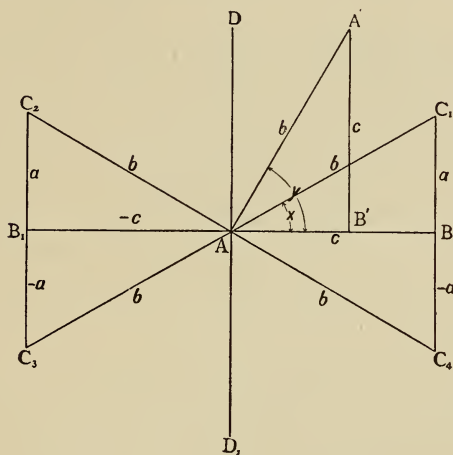


FIG. 17.

$B'$ , and  $A$  the position  $A'$ . Then the ordinate of  $A'$  will equal the abscissa of  $C_1$ , and the abscissa of  $A'$  will equal the ordinate of  $C_1$ . The three reciprocal ratios are:

*First.* The ratio of the distance of  $C_1$  (distance of  $A'$ ) to the ordinate of  $C_1$  (abscissa of  $A'$ ) is the secant of  $B'AA'$  ( $= C_1 = y$ ).

*Second.* The ratio of the abscissa of  $C_1$  (ordinate of  $A'$ ) to the ordinate of  $C_1$  (abscissa of  $A'$ ) is the tangent of  $y$ .

*Third.* The ratio of the abscissa of  $C_1$  (ordinate of  $A'$ ) to the distance of  $C_1$  (distance of  $A'$ ) is the sine of  $y$ .

Now the angle  $C_1 (= BAA' = y)$  is the complement of the angle  $BAC_1 (= x)$ ; that is,  $y = 90^\circ - x$ ; on this account any function of  $y$  is called the function of the complement of  $x$  or the complement's function, or simply **cofunction** of  $x$ .\*

Although the prefix co was obtained by thinking of  $y$ , and therefore  $x$  as being acute, the functions of  $y$  or the cofunctions of  $x$  will be defined as applying to all angles.

**The co-named functions.**—For any point on the terminal line of an angle:

The ratio of the distance to the ordinate is called the **cosecant** of the angle. The cosecant of  $x$  is written  $\csc x$ .

The ratio of the abscissa to the ordinate is called the **cotangent** of the angle. The cotangent of  $x$  is written  $\cot x$ .

The ratio of the abscissa to the distance is called the **cosine** of the angle. The cosine of  $x$  is written  $\cos x$ .

When the group of functions,  $\csc x$ ,  $\cot x$ ,  $\cos x$ , are referred to, they will be spoken of as the co-named functions of  $x$ .

But it must be noticed that  $\sin x$ ,  $\tan x$ ,  $\sec x$  are cofunctions; that is,  $\sin x$  is the cofunction of  $\cos x$ , just as  $\cos x$  is the cofunction of  $\sin x$ . In the same way  $\tan x$  and  $\cot x$  are cofunctions of each other;  $\sec x$  and  $\csc x$  are also cofunctions of each other.

---

\* Note that the complement of the complement of an angle is the angle itself [ $90^\circ - (90^\circ - x) = x$ ], and that any co-cofunction of the angle would be the function itself. .

## RELATIONS BETWEEN FUNCTIONS AND SEGMENTS.

11. In each quadrant

$$\frac{\text{ordinate of } C}{\text{distance of } C} \times \frac{\text{distance of } C}{\text{ordinate of } C} = 1.$$

$$\left. \begin{array}{l} \text{That is,} \quad \sin x \csc x = 1 \\ \text{Similarly,} \quad \tan x \cot x = 1 \\ \quad \quad \quad \sec x \cos x = 1 \end{array} \right\} \quad (1)$$

Therefore  $\sin x$  is the reciprocal of  $\csc x$ ,  $\csc x$  is the reciprocal of  $\sin x$ ; for this reason,  $\sin x$  and  $\csc x$  will be called **reciprocal functions** of  $x$ ; similar statements may be made of the other functions in group (1).

In each quadrant,

$$\frac{\text{ordinate of } C}{\text{distance of } C} = \frac{\text{ordinate of } C}{\text{abscissa of } C} \times \frac{\text{abscissa of } C}{\text{distance of } C}.$$

$$\left. \begin{array}{l} \text{That is,} \quad \sin x = \tan x \cos x \\ \text{Similarly,} \quad \tan x = \sec x \sin x \\ \quad \quad \quad \sec x = \csc x \tan x \\ \quad \quad \quad \csc x = \cot x \sec x \\ \quad \quad \quad \cot x = \cos x \csc x \\ \quad \quad \quad \cos x = \sin x \cot x \end{array} \right\} \quad (2)$$

Solving the equations in group (2) for each function,

$$\left. \begin{array}{l} \sin x = \frac{\tan x}{\sec x} = \frac{\cos x}{\cot x}; \quad \csc x = \frac{\cot x}{\cos x} = \frac{\sec x}{\tan x} \\ \tan x = \frac{\sec x}{\csc x} = \frac{\sin x}{\cos x}; \quad \cot x = \frac{\cos x}{\sin x} = \frac{\csc x}{\sec x} \\ \sec x = \frac{\csc x}{\cot x} = \frac{\tan x}{\sin x}; \quad \cos x = \frac{\sin x}{\tan x} = \frac{\cot x}{\csc x} \end{array} \right\} \quad (3)$$

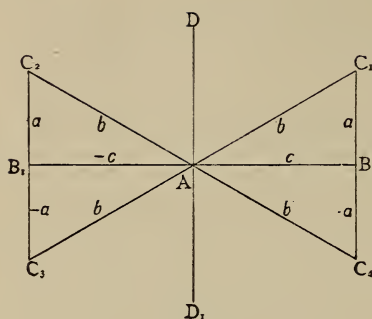


FIG. 18.

Value of each function.

If  $x$  is an angle of the first quadrant, the definitions give

$$\left. \begin{aligned} \sin x &= \frac{a}{b}; & \csc x &= \frac{b}{a} \\ \tan x &= \frac{a}{c}; & \cot x &= \frac{c}{a} \\ \sec x &= \frac{b}{c}; & \cos x &= \frac{c}{b} \end{aligned} \right\} \quad (4)$$

For the second quadrant,

$$\left. \begin{aligned} \sin x &= \frac{a}{b} = +\frac{a}{b}; & \csc x &= \frac{b}{a} = +\frac{b}{a} \\ \tan x &= \frac{a}{-c} = -\frac{a}{c}; & \cot x &= \frac{-c}{a} = -\frac{c}{a} \\ \sec x &= \frac{b}{-c} = -\frac{b}{c}; & \cos x &= \frac{-c}{b} = -\frac{c}{b} \end{aligned} \right\} \quad (5)$$

And similarly for the other two quadrants.

Values of the ordinate, distance, and abscissa of any point on the terminal line of an angle. — Clearing the equations in (4) of fractions gives for the first quadrant,

$$\left. \begin{aligned} a &= b \sin x = c \tan x \\ b &= c \sec x = a \csc x \\ c &= a \cot x = b \cos x \end{aligned} \right\} \quad (6)$$

The group in (5) gives for the second quadrant,

$$\left. \begin{aligned} a &= b \sin x = -c \tan x \\ b &= -c \sec x = a \csc x \\ c &= -a \cot x = -b \cos x \end{aligned} \right\} \quad (7)$$

And similarly for the other two quadrants.

**12. Device for memorizing relations.** — The diagram, Fig. 19, with the rules following, will be found useful in remembering the relations given in the groups of equations (1) to (7). In the figure, the angle  $x$  is drawn for the first quadrant; but if the proper algebraic signs of  $a$  and  $c$  are supplied, the diagram and all the rules apply to an angle of any quadrant.

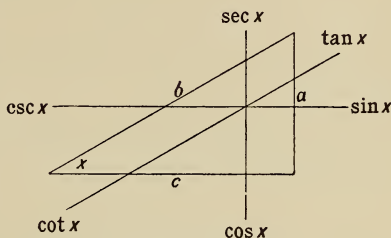


FIG. 19.

It will be seen that three cross-lines are drawn through  $a$  and  $b$ ,  $a$  and  $c$ ,  $b$  and  $c$ . Starting with the right angle and going in the positive direction of angles, the cross-lines are marked  $\sin x$ ,  $\tan x$ ,  $\sec x$ ,  $\csc x$ ,  $\cot x$ ,  $\cos x$ , respectively.

By going along the cross-lines, the relations in groups (4), (5), (6), and (7) are given by the two following rules:

**RULE I.** Any segment is equal to the product of the two adjacents; thus along the horizontal cross-line,  $a = b \sin x$  and  $b = a \csc x$ .

**RULE II.** Any segment or function is equal to the quotient of the adjacent by the next; thus along the horizontal cross-line,

$$\sin x = \frac{a}{b}, \quad a = \frac{b}{\csc x}, \quad b = \frac{a}{\sin x}, \quad \csc x = \frac{b}{a}.$$

By going around either way outside the triangle, considering only the functions, equations (2) and (3) are given by two rules:

**RULE III.** Any function is equal to the product of the two adjacents; thus  $\sin x = \tan x \cos x$ .

RULE IV. Any function is equal to the quotient of the adjacent by the next; thus

$$\sin x = \frac{\tan x}{\sec x} \quad \text{or} \quad \sin x = \frac{\cos x}{\cot x}.$$

Considering functions on the same cross-line, group (1) is given by

RULE V. Any function is equal to the reciprocal of its opposite function; thus  $\sin x \csc x = 1$ .

### ABSOLUTE VALUES AND ALGEBRAIC SIGNS OF FUNCTIONS.

**13. DEFINITIONS.** The *numerical value* of a magnitude is the ratio of the magnitude to another magnitude of the same kind taken as a unit. The *absolute value* of a magnitude is the numerical value of the magnitude considered without regard to sign. Thus the numerical value of  $-5$  feet is  $-5$ , and the absolute value is simply  $5$ .

**Absolute values of functions of angles in terms of functions of angles between  $0^\circ$  and  $90^\circ$ .**—In Fig. 17, denote the positive acute value of angle  $BAC_1$  by  $x$ . Let the least positive value of angle  $BAC_2 = 180^\circ - x$ , of angle  $BAC_3 = 180^\circ + x$ , and of angle  $BAC_4 = 360^\circ - x$ . Also make the distances  $AC_1, AC_2, AC_3, AC_4$  all equal. Then the triangles  $BAC_1, B_1AC_2, B_1AC_3, BAC_4$  are all equal.

When the revolving half-line is at  $B$  or  $B_1$ , the angles generated from  $AB$  are all included in the expression  $n$  times  $90^\circ$ , where  $n$  is an even whole number. And when the revolving half-line is at  $D$  or  $D_1$ , the angles generated from  $AB$  are all included in the expression  $n$  times  $90^\circ$ , where  $n$  is an odd number. Hence the angles generated when the terminal line is at  $AC_1, AC_2, AC_3$ , or  $AC_4$  are included in the expressions  $90n \pm x$  where  $n$  is even, and  $90n \pm y$  where  $n$  is odd.

From the equality of the various triangles the ordinates and abscissæ of  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  are all equal in absolute value. Hence the functions of  $BAC_1$ ,  $BAC_2$ ,  $BAC_3$ ,  $BAC_4$

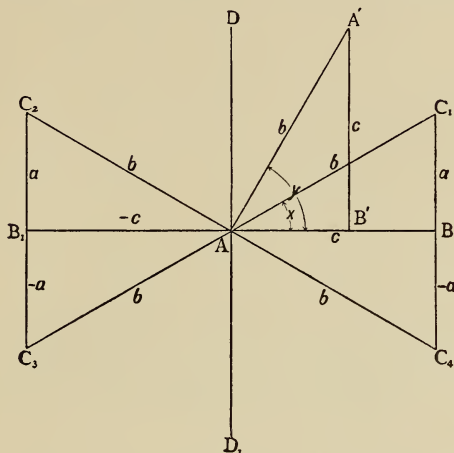


FIG. 17.

are equal in absolute value to the same functions of  $x$ ; and therefore also equal to the cofunctions of  $y$ . That is:

I. *The absolute value of any function of an angle  $90n \pm x$  where  $n$  is even, is the same as the absolute value of that function of  $x$ .*

II. *The absolute value of any function of an angle  $90n \pm y$  where  $n$  is odd is the same as the absolute value of that cofunction of  $y$ .*

Expressed in a formula,  $f$  standing for any function:

$f(90n \pm x) = \pm fx$  when  $n$  is even  $= \pm \text{cofx}$  when  $n$  is odd.

For example: Any function of  $150^\circ$ ,  $210^\circ$ ,  $330^\circ$ ,  $390^\circ$ ,  $510^\circ$ ,  $-30^\circ$ ,  $-150^\circ$ , etc., is equal in absolute value to the same function of  $30^\circ$  or the co-named function of  $60^\circ$ .

The algebraic signs of the functions may be remembered by this device :

Write the functions in the order

|     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|
| sin | tan | sec | csc | cot | cos |
| 2   | 3   | 4   | 2   | 3   | 4   |

The functions with positive values are :

In Quadrant I. All of them.

In Quadrant II. sin csc

In Quadrant III. tan cot

In Quadrant IV. sec cos

In all other cases the functions are negative in value.

The method of obtaining the value of a function of any angle in terms of an angle between  $0^\circ$  and  $90^\circ$  is given in these two rules :

**RULE I.** Divide the angle by  $90^\circ$ . If the quotient is even, take the same function of the remainder. If the quotient is odd, take the cofunction of the remainder.

**RULE II.** Prefix the proper algebraic sign.

$$\text{Thus } \sin 172^\circ = \cos 82^\circ. \quad \csc 274^\circ 9' = -\sec 4^\circ 9'$$

$$\tan 305^\circ = -\cot 35^\circ. \quad \cot 329^\circ 51' = -\tan 59^\circ 51'$$

$$\sec 595^\circ = -\sec 55^\circ. \quad \cos 935^\circ 22' = -\cos 35^\circ 22'$$

Notice that the quadrant is indicated by 1 + the quotient.

**14. Functions of  $180^\circ \pm x$ , of  $90^\circ + x$ , and of  $-x$ .**—Consider any function, say the sine, of the two angles  $180^\circ - x$  and  $x$ . It has been shown that  $\sin(180^\circ - x)$  and  $\sin x$  are equal in absolute value. Consider now their algebraic signs.

If  $x$  is of the first quadrant,  $180^\circ - x$  is of the second quadrant; then  $\sin x$  and  $\sin(180^\circ - x)$  are both +.

If  $x$  is of the second quadrant,  $180^\circ - x$  is of the first quadrant; then  $\sin x$  and  $\sin(180^\circ - x)$  are both  $+$ .

If  $x$  is of the third quadrant,  $180^\circ - x$  is of the fourth quadrant; then  $\sin x$  and  $\sin(180^\circ - x)$  are both  $-$ .

If  $x$  is of the fourth quadrant,  $180^\circ - x$  is of the third quadrant; then  $\sin x$  and  $\sin(180^\circ - x)$  are both  $-$ .

Hence the equation  $\sin(180^\circ - x) = \sin x$  is true for all values of  $x$ .

In a similar manner, it may be shown that the equation  $\sin(90^\circ + x) = \cos x$  is true for all values of  $x$ . Also that  $\sin(180^\circ + x) = \sin(-x) = -\sin x$  are true for all values of  $x$ .

In a similar manner, other functions of these angles may be considered. The results are shown in the following table:

|                                  |                                 |
|----------------------------------|---------------------------------|
| $\sin(180^\circ - x) = \sin x.$  | $\sin(90^\circ + x) = \cos x.$  |
| $\tan(180^\circ - x) = -\tan x.$ | $\tan(90^\circ + x) = -\cot x.$ |
| $\sec(180^\circ - x) = -\sec x.$ | $\sec(90^\circ + x) = -\csc x.$ |
| $\csc(180^\circ - x) = \csc x.$  | $\csc(90^\circ + x) = \sec x.$  |
| $\cot(180^\circ - x) = -\cot x.$ | $\cot(90^\circ + x) = -\tan x.$ |
| $\cos(180^\circ - x) = -\cos x.$ | $\cos(90^\circ + x) = -\sin x.$ |
| $\sin(180^\circ + x) = -\sin x.$ | $\sin(-x) = -\sin x.$           |
| $\tan(180^\circ + x) = \tan x.$  | $\tan(-x) = -\tan x.$           |
| $\sec(180^\circ + x) = -\sec x.$ | $\sec(-x) = \sec x.$            |
| $\csc(180^\circ + x) = -\csc x.$ | $\csc(-x) = -\csc x.$           |
| $\cot(180^\circ + x) = \cot x.$  | $\cot(-x) = -\cot x.$           |
| $\cos(180^\circ + x) = -\cos x.$ | $\cos(-x) = \cos x.$            |

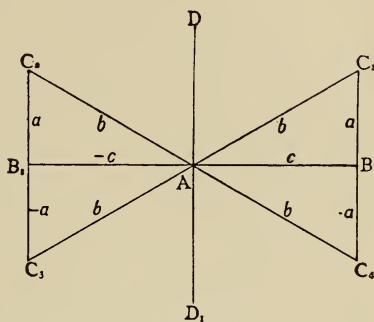


FIG. 18.

## THE CURVE OF EACH FUNCTION.

**15.** If a point moves so that its distance remains the same, the ordinate of the moving point will be proportional to the sine of the varying angle; and the abscissa of the moving point will be proportional to the cosine of the angle; for the sine of the angle equals the ordinate divided by the distance; and the cosine of the angle equals the abscissa divided by the distance; and therefore if the distance remains the same, the sine varies as the ordinate; and the cosine varies as the abscissa.

Suppose, now, a second point to move in the plane so that its abscissa is proportional to the angle, and its ordinate is proportional to the sine of that angle. As the angle changes from  $0^\circ$  to  $360^\circ$ , this second point will trace out the

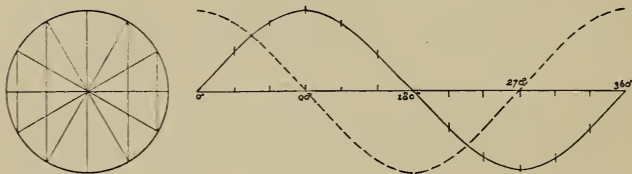


FIG. 20. — Curve of sines.

Curve of cosines. - - - -

curve shown in Fig. 20, called a simple harmonic curve, or **curve of sines**. Or, if the ordinate of the second point was proportional to the cosine of the angle, the curve is called the **curve of cosines**.

If the first point moves in a line at right angles to the initial line, the ordinate of this point will be proportional to the tangent of the angle, and the distance of this moving point will be proportional to the secant of the angle; now if the abscissa of the second moving point was proportional to the angle and the ordinate proportional to the tangent of the angle, the curve traced out is called the **curve of tangents** (Fig. 21); or, if the ordinate of the second point

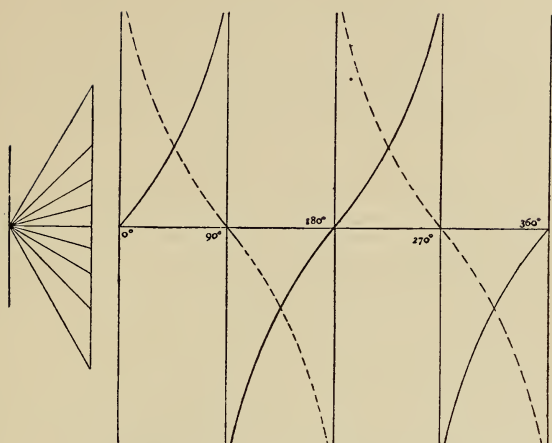


FIG. 21.—Curve of tangents. — Curve of cotangents. - - - -

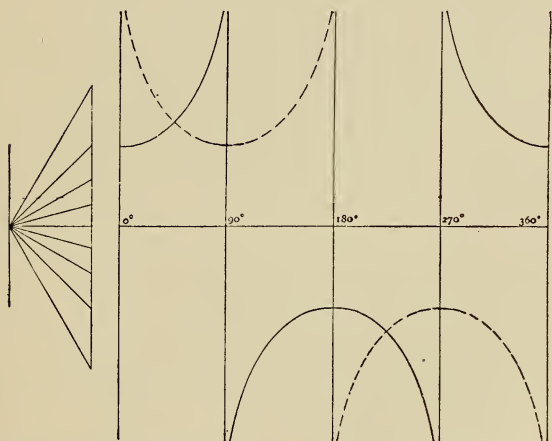


FIG. 22.—Curve of secants. — Curve of cosecants. - - - -

was proportional to the secant of the angle, the curve is called the **curve of secants** (Fig. 22).

In a similar manner, the first point may be assumed to move parallel to the initial line, and the second point may be made to trace the **curve of cosecants** (Fig. 22), or the **curve of cotangents** (Fig. 21).

#### LIMITING VALUES OF FUNCTIONS.

**16.** Since from geometry the hypotenuse of a right triangle can never be less than either side, the distance of a point can never be less than either the abscissa or ordinate of the point. Therefore the absolute value of the sine or of the cosine of any angle is never more than 1, and the absolute value of the secant or of the cosecant of any angle is never less than 1. The absolute value of the tangent or of the cotangent of angles is without limit.

#### THE ANGLES EACH LESS THAN $360^\circ$ CORRESPONDING TO A GIVEN VALUE OF A FUNCTION.\*

**17.** From what has been said, it follows that there is but one value of any function of a given angle; but for a given numerical value of any function of the angle, there will be 0, 1, or 2 values of the angle between  $0^\circ$  and  $360^\circ$ .

1. If  $\sin x$  or  $\cos x$  is more than 1 or less than  $-1$ , or if  $\sec x$  or  $\csc x$  is between 1 and  $-1$ ,  $x$  will be impossible.

2. If  $\sin x = +1$  or  $\csc x = +1$ ,  $x = 90^\circ$ .

If  $\sin x = -1$  or  $\csc x = -1$ ,  $x = 270^\circ$ .

If  $\sec x = +1$  or  $\cos x = +1$ ,  $x = 0^\circ$ .

If  $\sec x = -1$  or  $\cos x = -1$ ,  $x = 180^\circ$ .

3. In all other cases, two values of  $x$  between  $0^\circ$  and  $360^\circ$  will be obtained.

---

\* This section can be more easily followed if the diagrams, Figs. 20, 21, and 22, are referred to.

Suppose that  $\sin x = N$  (or  $\csc x = N$ ), and that  $x_1$  is the corresponding acute value of  $x$ . Then  $x$  also equals  $180^\circ - x_1$ .

If the given equation is  $\sin x = -N$  (or  $\csc x = -N$ ) and  $\sin x_1 = N$  (or  $\csc x_1 = N$ ), where  $x_1$  is acute, then

$$x = 180^\circ + x_1 \text{ or } x = 360^\circ - x_1.$$

If  $\tan x = N$  (or  $\cot x = N$ ), and  $x_1$  is the acute value of  $x$ , another value of  $x$  is  $180^\circ + x_1$ .

If the given equation is  $\tan x = -N$  (or  $\cot x = -N$ ) and  $\tan x_1 = N$  (or  $\cot x_1 = N$ ) where  $x_1$  is acute, then

$$x = 180^\circ - x_1 \text{ or } x = 360^\circ - x.$$

If  $\sec x = N$  (or  $\cos x = N$ ) and  $x_1$  is the acute value of  $x$ , another value of  $x$  is  $360^\circ - x_1$ .

If the given equation is  $\sec x = -N$  (or  $\cos x = -N$ ) and  $\sec x_1 = N$  (or  $\cos x_1 = N$ ) where  $x_1$  is acute, then

$$x = 180^\circ - x_1 \text{ or } x = 180^\circ + x_1.$$

**Angle less than  $180^\circ$ .**—If the problem of obtaining the values of an angle from a given function of the angle is still further limited by the condition that the angle shall be less than  $180^\circ$  (say an angle of a triangle), then :

(1) If the given function is a sine or a cosecant with a negative value, no value of the angle could be obtained; because all angles between  $0^\circ$  and  $180^\circ$  have a positive sine and a positive cosecant.

(2) If the given function is a sine or a cosecant with a positive value, and the given value of the function was possible, there will be two supplementary values of the angle; because if  $x_1$  is the acute value,  $180^\circ - x_1$  will be the obtuse value. If  $x_1$  happens to be  $90^\circ$ , it may be said that the two values are equal, or there is but one value.

(3) If the given function of the required angle is any other function than the sine or cosecant, there can be but

one value of the angle less than  $180^\circ$ ; acute if the value of the function is positive; obtuse if the value of the function is negative.

If it is known that a required angle was **not greater than**  $90^\circ$ , a negative value of any function of the angle would indicate that no value of the angle was possible.

### FUNCTIONS OF $\frac{1}{2}x$ AND $2x$ .

**18. Functions of  $\frac{1}{2}x$ .** — Bisect angle  $BAC$ , and draw  $BE$  parallel to  $AD$ . Denote angle  $BAC$  by  $x$ . Then angle  $E = \text{angle } ABE = \text{angle } BAD = \frac{1}{2}x$ . Also  $AE = AB$ .

$BC^2 = AC^2 - AB^2$ . Divide by  $BC^2$ .

$$1 = \csc^2 x - \cot^2 x. \quad (8)$$

$$\frac{AB}{BD} = \frac{EA}{BD} = \frac{EC}{BC} = \frac{AC}{BC} + \frac{AB}{BC}.$$

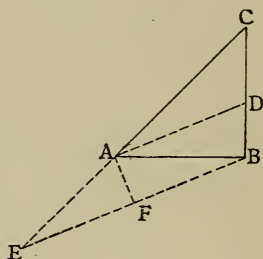


FIG. 23.

$$\cot \frac{1}{2}x = \csc x + \cot x = \frac{1 + \cos x}{\sin x}. \quad (9)$$

$$\frac{EB}{AD} = \frac{EC}{AC} \text{ and } \frac{BD}{EA} = \frac{BC}{EC}; \text{ whence } \frac{EB}{AE} \times \frac{BD}{AD} = \frac{BC}{AC}.$$

$$2 \cos \frac{1}{2}x \sin \frac{1}{2}x = \sin x. \quad (10)$$

The quotient of (8) and (9) is

$$\tan \frac{1}{2}x = \csc x - \cot x = \frac{1 - \cos x}{\sin x}. \quad (11)$$

The product of (10) and (11) is

$$2 \sin^2 \frac{1}{2}x = 1 - \cos x. \quad (12)$$

The product of (9) and (10) is

$$2 \cos^2 \frac{1}{2}x = 1 + \cos x. \quad (13)$$

The half difference of (13) and (12) is

$$\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x = \cos x. \quad (14)$$

The reciprocal of (10) is

$$\frac{1}{2} \sec \frac{1}{2} x \csc \frac{1}{2} x = \csc x. \quad (15)$$

**Functions of  $2x$ .**—Denote now the angle  $BAD$  by  $x$ . Then from (10), (14), and (15):

$$\sin 2x = 2 \sin x \cos x. \quad (16)$$

$$\cos 2x = \cos^2 x - \sin^2 x. \quad (17)$$

$$\csc 2x = \frac{1}{2} \sec x \csc x. \quad (18)$$

The other functions can then be found in terms of these.

### FUNCTIONS OF PARTICULAR ANGLES.

**19. Functions of  $45^\circ$ ,  $22^\circ 30'$ , etc.**—Let angle  $A = 45^\circ$ . Denote the length of  $AB$  by  $c$ ; then  $BC = c$  and  $AC = \sqrt{c^2 + c^2} = 1.4142 c$ . Hence

$$\tan 45^\circ = \cot 45^\circ = 1.$$

$$\sec 45^\circ = \csc 45^\circ = 1.4142.$$

Also,  $\sin 45^\circ = \cos 45^\circ = .7071.$

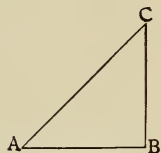


FIG. 24.

From these values of the functions of  $45^\circ$ , the values of the functions of  $22^\circ 30'$  can be found by means of formulæ (9) to (13). Then from the values of the functions of  $22^\circ 30'$ , the functions of  $11^\circ 15'$  can be found; and so on.

**Functions of  $60^\circ$ ,  $30^\circ$ ,  $15^\circ$ , etc.**—The triangle having angles of  $90^\circ$ ,  $60^\circ$ , and  $30^\circ$  is one-half of an equilateral triangle.

Take  $BC = a$ ; then  $AC = 2a$  and

$$AB = \sqrt{4a^2 - a^2} = 1.7321 a.$$

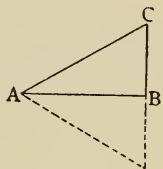


FIG. 25.

Hence the functions of  $60^\circ$  and  $30^\circ$  can be computed. And, therefore, the functions of  $15^\circ$ ,  $7^\circ 30'$ , etc., may be computed.

**Functions of  $72^\circ$ ,  $36^\circ$ ,  $18^\circ$ , etc.**—Let  $AC = CD$ , and  $DE$  bisect angle  $D$ ,  $CB$  bisect angle  $C$ ; take the value of angle  $A$  to be  $72^\circ$ . All the other angles on the figure can then be found. From the equality of angles,  $AD = ED = EC$ ; also the triangle  $ACD$  is similar to the triangle  $ADE$ .

Denote the length of  $AC$  by  $b$ , and the length of  $AB = b \cos 72^\circ$  by  $bx$ . Then  $AE = b - 2bx$ .

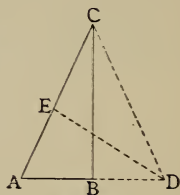


FIG. 26.

From the similar triangles  $ACD$  and  $ADE$ ,

$$\frac{AC}{AD} = \frac{AD}{AE} \quad \text{or} \quad \frac{b}{2bx} = \frac{2bx}{b - 2bx}.$$

Solving:  $x = \cos 72^\circ = \sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1) = .3090$ .

Denote the length of  $BC$  by  $by$ ; then

$$BC^2 = AC^2 - AB^2 \quad \text{or} \quad b^2y^2 = b^2 - b^2x^2.$$

That is,  $y^2 = 1 - x^2$  or  $\sin^2 72^\circ = \cos^2 18^\circ = 1 - .3090^2$ .

Hence the other functions of  $72^\circ$  and  $18^\circ$  may be found. Also the functions of  $36^\circ$ ,  $9^\circ$ ,  $4^\circ 30'$ , etc., may be found.

**Exercises.**—For all values of  $x$ , prove that

$$\sin^2 x + \cos^2 x = 1,$$

$$\sec^2 x - \tan^2 x = 1,$$

$$\csc^2 x - \cot^2 x = 1,$$

and explain how all the functions of an angle may be found when the value of only one function is known.

## CHAPTER II.

### EXPLANATION OF THE TABLES.

#### I. NATURAL VALUES OF FUNCTIONS.

**20.** The values given for the functions of angles in The Table of Natural Values of Functions, page i to page vii, are approximate values to four significant figures at least. The last figure given in each value is the nearest figure in that place. Thus the value of  $\sin 32^\circ$ , which is not less than .52985 and not greater than .52995, is given on page v, line 25, as .5299.

Any value printed without a decimal point will have one or more figures missing. The decimal point and the missing figures will be found to the left and slightly above the figures printed for the value. Thus page v, line 29,  $\sin 32^\circ 40'$  is .5398, the 398 being printed in the proper place and .5 being found higher and to the left.

$$\tan 32^\circ 40' = .6412, \sec 32^\circ 40' = 1.1879, \text{ etc.}$$

**Given an angle to find any function of the angle.**—Angles between  $0^\circ$  and  $45^\circ$  will be found in the column marked  $x$  on the left of the page, and angles between  $45^\circ$  and  $90^\circ$  in the column marked  $y$  on the right. When the angle is found on the left, the names of the functions are given at the top of the columns,  $x$  being added to the name of each function. When the angle is found on the right, the names of the functions (marked  $y$ ) are given at the bottom of the page; thus:

|                   |                                |
|-------------------|--------------------------------|
| Page v, line 39   | $\sin 34^{\circ} 20' = .5640$  |
| Page vi, line 9   | $\tan 53^{\circ} 40' = 1.3597$ |
| Page iv, line 2   | $\sec 21^{\circ} 10' = 1.0723$ |
| Page vii, line 16 | $\csc 45^{\circ} 30' = 1.4020$ |
| Page iv, line 26  | $\cot 64^{\circ} 50' = .4699$  |
| Page iv, line 7   | $\cos 22^{\circ} = .9272$      |

To find function of angles which are not multiples of 10'. — When we desire the function of an angle, whose value is between two angles of the table, a process called **interpolation** is used.

By reference to any part of the table, it will be seen that **small changes in the angle produce nearly proportional changes in the values of any function of the angle.** A four-place table usually gives the values of the functions of angles for every 10'. By assuming that the change in any function is proportional to small changes in the angle, the value of the function of an angle to a single minute can in general be found to four significant figures without appreciable error. Conversely, if the value of the function is correct to four significant figures, the value of the angle can in general be found correct to a single minute.

A five-place table giving the values of the functions for each minute can be used to interpolate to 10"; a six-place table for every 10" can be interpolated to 1"; a seven-place table for every second can be interpolated to .1", and so on.

With every increase in the number of significant figures, the degree of precision is increased and also the labor of using the tables. If the data of a computation contain angles measured to a single minute, four-place tables can be used in the computation; if the angles are measured to one-tenth second, seven-place tables can be used; and so on.

Let  $f$  stand for any function. Then if  $h$  is a small addition to angle  $x$ , and  $z$  is the change in any function of  $x$  due to a change  $k$  ( $10'$ ) in the angle, it is assumed

$$\frac{f(x+h) - fx}{z} = \frac{h}{k}$$

or 
$$f(x+h) = fx + \frac{h}{k}z,$$

from which the value of a function of an angle  $x+h$  can be obtained by substitution. Thus:

$$\begin{aligned}\sin 34^\circ 23' &= \sin 34^\circ 20' + .3 (\sin 34^\circ 30' - \sin 34^\circ 20') \\ &= .5640 + .3 \times .0024 = .5647.\end{aligned}$$

$$\begin{aligned}\tan 53^\circ 44' &= \tan 53^\circ 40' + .4 (\tan 53^\circ 50' - \tan 53^\circ 40') \\ &= 1.3597 + .4 \times .0083 = 1.3630.\end{aligned}$$

$$\begin{aligned}\sec 21^\circ 11' &= \sec 21^\circ 10' + .1 (\sec 21^\circ 20' - \sec 21^\circ 10') \\ &= 1.0723 + .1 \times .0013 = 1.0724.\end{aligned}$$

$$\begin{aligned}\csc 45^\circ 38' &= \csc 45^\circ 30' + .8 (\csc 45^\circ 40' - \csc 45^\circ 30') \\ &= 1.4020 + .8 \times (-.0040) = 1.3988.\end{aligned}$$

$$\begin{aligned}\cot 64^\circ 55' &= \cot 64^\circ 50' + .5 (\cot 65^\circ - \cot 64^\circ 50') \\ &= .4699 + .5 (-.0036) = .4681.\end{aligned}$$

$$\begin{aligned}\cos 22^\circ 7' &= \cos 22^\circ + .7 (\cos 22^\circ 10' - \cos 22^\circ) \\ &= .9272 + .7 (-.0011) = .9264.\end{aligned}$$

It should be noted that all of the co-named functions of acute angles decrease as the angle increases. Hence when the tabulated value of a co-named function of an acute angle is corrected for an addition to the angle, the correction will be negative.

If functions of angles greater than  $90^\circ$  are desired, they are obtained by applying the rules on page 22; thus (Quadrant II),

$$\sin 155^{\circ} 12' = \cos 65^{\circ} 12' = .4195.$$

$$\tan 113^{\circ} 47' = -\cot 23^{\circ} 47' = -2.2691.$$

$$\sec 167^{\circ} 9' = -\csc 77^{\circ} 9' = -1.0257.$$

$$\csc 123^{\circ} 42' = \sec 33^{\circ} 42' = 1.2020.$$

$$\cot 147^{\circ} 56' = -\tan 57^{\circ} 56' = -1.5962.$$

$$\cos 134^{\circ} 35' = -\sin 44^{\circ} 35' = -.7019.$$

And in a similar manner for angles of Quadrants III and IV.

**Given the value of a function of the angle, to find the value of that angle.** — The table is arranged to give the value of the function when the angle is known. The reverse operation of obtaining the angle when the function is known is somewhat more inconvenient. Assume that we know the sine of the required angle; the given value may be sought either in the column marked **sin *x*** at the top of the page or in the column marked **sin *y*** at the bottom of the page. If the number is found in the column marked **sin *x***, and it is known that the angle is acute, the value of the angle will be found in the column marked ***x*** on the left of the page. If the given value of the sine is found in the column marked **sin *y***, the value of the acute angle will be found in the column marked ***y***. Any other function will be found in a similar manner; thus:

Page v, line 27. If  $\sin A = .8450$ ,  $A$  (acute)  $= 57^{\circ} 40'$ .

Page v, line 4. If  $\tan A = 1.8418$ ,  $A$  (acute)  $= 61^{\circ} 30'$ .

Page v, line 41. If  $\sec A = 1.2158$ ,  $A$  (acute)  $= 34^{\circ} 40'$ .

Page iv, line 33. If  $\csc A = 2.2543$ ,  $A$  (acute)  $= 26^{\circ} 20'$ .

Page v, line 34. If  $\cot A = .6619$ ,  $A$  (acute)  $= 56^{\circ} 30'$ .

Page vi, line 27. If  $\cos A = .7735$ ,  $A$  (acute)  $= 39^{\circ} 20'$ .

If the values of the functions of  $45^{\circ}$  are memorized

$$(\sin 45^{\circ} = .7, \tan 45^{\circ} = 1, \sec 45^{\circ} = 1.4),$$

it will be known whether the given value of the functions should be sought in the columns on the right of the page or on the left. Thus if  $\sin A = .9112$ , the value of  $A$  must be more than  $45^\circ$ , because the sine of an acute angle increases as the angle increases, and the sine of  $45^\circ$  is  $.7$ . Since the angle is more than  $45^\circ$ , the given value is sought in the column marked  $\sin y$ .

As before, let  $f$  stand for any function,  $h$  a small addition to angle  $x$ , and  $z$  a change in any function due to a change  $k(10')$  in the angle. Then, since small changes in the angle are nearly proportional to the changes in any function,

$$\frac{h}{k} = \frac{f(x+h) - fx}{z},$$

$$x + h = x + \frac{f(x+h) - fx}{z} k.$$

If  $\sin A = .6479$ , the acute value of  $A$

$$= 40^\circ 20' + \frac{.6479 - .6472}{.6494 - .6472} 10' = 40^\circ 23'.$$

If  $\tan A = 3.7539$ , the acute value of  $A$

$$= 75^\circ + \frac{3.7539 - 3.7321}{3.7760 - 3.7321} 10' = 75^\circ 5'.$$

If  $\sec A = 1.2190$ , the acute value of  $A$

$$= 34^\circ 50' + \frac{1.2190 - 1.2183}{1.2208 - 1.2183} 10' = 34^\circ 53'.$$

If  $\csc A = 3.9984$ , the acute value of  $A$

$$= 14^\circ 20' + \frac{3.9984 - 4.0394}{3.9939 - 4.0394} 10' = 14^\circ 29'.$$

If  $\cot A = 2.2062$ , the acute value of  $A$

$$= 24^\circ 20' + \frac{2.2062 - 2.2113}{2.1943 - 2.2113} 10' = 24^\circ 23'.$$

If  $\cos A = .5654$ , the acute value of  $A$

$$= 55^\circ 30' + \frac{.5654 - .5664}{.5640 - .5664} 10' = 55^\circ 34'.$$

If both values of the angle which are less than  $360^\circ$  are desired, any value, not acute, can be obtained by a reversal of the rules on page 22; thus,

$$\text{If } \sin A = .6479, \quad A = 40^\circ 23' \text{ or } 139^\circ 37'.$$

$$\text{If } \tan A = 3.7539, \quad A = 75^\circ 5' \text{ or } 255^\circ 5'.$$

$$\text{If } \sec A = 1.2190, \quad A = 34^\circ 53' \text{ or } 325^\circ 7'.$$

$$\text{If } \csc A = 3.9984, \quad A = 14^\circ 29' \text{ or } 165^\circ 31'.$$

$$\text{If } \cot A = 2.2062, \quad A = 24^\circ 23' \text{ or } 204^\circ 23'.$$

$$\text{If } \cos A = .5654, \quad A = 55^\circ 34' \text{ or } 304^\circ 26'.$$

Also when the given function is negative:

$$\text{If } \sin A = - .6479, \quad A = 220^\circ 23' \text{ or } 319^\circ 37'.$$

$$\text{If } \tan A = - 3.7539, \quad A = 104^\circ 55' \text{ or } 284^\circ 55'.$$

$$\text{If } \sec A = - 1.2190, \quad A = 145^\circ 7' \text{ or } 214^\circ 53'.$$

$$\text{If } \csc A = - 3.9984, \quad A = 194^\circ 29' \text{ or } 345^\circ 31'.$$

$$\text{If } \cot A = - 2.2062, \quad A = 155^\circ 37' \text{ or } 335^\circ 37'.$$

$$\text{If } \cos A = - .5654, \quad A = 124^\circ 26' \text{ or } 235^\circ 34'.$$

It should be noted that if the given value of the sine or cosine of an angle is greater than  $+1$  or less than  $-1$ , no value of the angle can be obtained; and if the given value of the secant or cosecant of an angle is between  $+1$  and  $-1$ , no value of the angle can be obtained.

**Exercises.** — Copy one value in any row of the following table. Then find each of the remaining values in the row.

|                      |                      |        |                      |                      |
|----------------------|----------------------|--------|----------------------|----------------------|
| $\sin 0^\circ 43'$   | $\sin 179^\circ 17'$ | .0125  | $\cos 89^\circ 17'$  | $\cos 270^\circ 43'$ |
| $\sin 10^\circ 18'$  | $\sin 169^\circ 42'$ | .1788  | $\cos 79^\circ 42'$  | $\cos 280^\circ 18'$ |
| $\sin 200^\circ 41'$ | $\sin 339^\circ 19'$ | -.3532 | $\cos 110^\circ 41'$ | $\cos 249^\circ 19'$ |
| $\sin 210^\circ 6'$  | $\sin 329^\circ 54'$ | -.5015 | $\cos 120^\circ 6'$  | $\cos 239^\circ 54'$ |
| $\sin 40^\circ 23'$  | $\sin 139^\circ 37'$ | .6479  | $\cos 49^\circ 37'$  | $\cos 310^\circ 23'$ |

|                      |                      |          |                      |                      |
|----------------------|----------------------|----------|----------------------|----------------------|
| $\tan 84^\circ 15'$  | $\tan 264^\circ 15'$ | 9.9310   | $\cot 5^\circ 45'$   | $\cot 185^\circ 45'$ |
| $\tan 74^\circ 18'$  | $\tan 254^\circ 18'$ | 3.5576   | $\cot 15^\circ 42'$  | $\cot 195^\circ 42'$ |
| $\tan 157^\circ 13'$ | $\tan 337^\circ 13'$ | — .4200  | $\cot 112^\circ 47'$ | $\cot 292^\circ 47'$ |
| $\tan 141^\circ 33'$ | $\tan 321^\circ 33'$ | — .7940  | $\cot 128^\circ 27'$ | $\cot 308^\circ 27'$ |
| $\tan 135^\circ 19'$ | $\tan 315^\circ 19'$ | — .9890  | $\cot 134^\circ 41'$ | $\cot 314^\circ 41'$ |
| $\sec 22^\circ 34'$  | $\sec 337^\circ 26'$ | 1.0829   | $\csc 67^\circ 26'$  | $\csc 112^\circ 34'$ |
| $\sec 63^\circ 12'$  | $\sec 296^\circ 48'$ | 2.2179   | $\csc 26^\circ 48'$  | $\csc 153^\circ 12'$ |
| $\sec 99^\circ 11'$  | $\sec 260^\circ 49'$ | — 6.2659 | $\csc 189^\circ 11'$ | $\csc 350^\circ 49'$ |
| $\sec 128^\circ 43'$ | $\sec 231^\circ 17'$ | — 1.5988 | $\csc 218^\circ 43'$ | $\csc 321^\circ 17'$ |
| $\sec 107^\circ 47'$ | $\sec 252^\circ 13'$ | — 3.2742 | $\csc 197^\circ 47'$ | $\csc 342^\circ 13'$ |

## II. LOGARITHMS OF NUMBERS.

**21. Definition.** — Logarithms are a series of numbers severally assigned to the ordinary numbers, and so arranged that the multiplication and division of ordinary numbers is accomplished by means of the addition or subtraction of their logarithms.

Many systems of logarithms are possible; but in the simplest systems the ordinary numbers are powers of a number called a **base**, and the logarithms are the corresponding **exponents**.

Thus, if the base is 2, the numbers are powers of 2, and the corresponding exponents are called logarithms to the base 2. To multiply 64 by 16, we add the logarithms of 64 and 16 (6 and 4), producing 10; and 10 is the logarithm of 1024, the product.

| NUMBERS<br>OR<br>POWERS. | LOGARITHMS<br>OR<br>EXPONENTS. |
|--------------------------|--------------------------------|
| 1                        | 0                              |
| 2                        | 1                              |
| 4                        | 2                              |
| 8                        | 3                              |
| 16                       | 4                              |
| 32                       | 5                              |
| 64                       | 6                              |
| 128                      | 7                              |
| 256                      | 8                              |
| 512                      | 9                              |
| 1024                     | 10                             |

To divide 512 by 32, subtract their logarithms (9 and 5), giving 4; and 4 is the logarithm of 16, the desired quotient.

To raise 4 to the fifth power, multiply the logarithm of 4 (2) by the index of the power (5), producing 10, the logarithm of 1024, the required power.

To obtain the cube root of 512, divide the logarithm of 512 (9) by the index of the root (3), giving 3, the logarithm of 8, the desired root.

Apply to logarithms the rules of algebra relating to exponents. Then we have

Log of product = sum of logs of factors.

Log of quotient = log of dividend - log of divisor.

Log of power of a number = log of number  $\times$  index of power.

Log of root of a number = log of number  $\div$  index of root.

#### CHARACTERISTIC AND MANTISSA.

If we think of integers as being arranged with zeros expressed decimally (thus,  $7 = 7.0000$ ), all logarithms may be said to consist of two parts,—an integral part and a fractional part.

The integral part of a logarithm is called its **characteristic**. The fractional part is called its **mantissa**. Thus, if 2.1347 is a logarithm, 2 is its characteristic, and .1347 is its mantissa.

The characteristic may be either positive or negative, but the logarithm is so arranged that the mantissa is always positive. Thus, if the logarithm was  $-2.4178$ , the number 3 would be added and subtracted thus,  $3 - 2.4178 - 3 = .5822 - 3$ . In this logarithm,  $-3$  is the characteristic, and .5822 is the mantissa.

**Exercises.** — Find the characteristic and mantissa of the following logarithms:  $4\frac{1}{2}$ ,  $-1\frac{3}{4}$ ,  $\frac{2}{3}$ ,  $-\frac{10}{9}$ ,  $-\frac{5}{6}$ .

## DENARY LOGARITHMS.

Although a practical table of logarithms could be constructed by taking any positive number as the base of the system, the **common system** of logarithms has 10 for its base.

The system whose base is 10 has two principal advantages:

(1) The **mantissa** of a logarithm depends only on the figures of the number and their order; or, to express it another way, the mantissa of the logarithm of a number is independent of the position of the decimal point in the number. For example, the numbers 4.57 and 4570 have the same mantissa in their logarithms.

(2) The **characteristic** of a logarithm depends only on the position from the units place of the first significant figure of the number; or, to express it another way, the characteristic of the logarithm of a number is independent of the figures in the number and of the order of the figures. For example, the numbers 45.7 and 23.8 have the same characteristic in their logarithms.

To prove the first of these principles, say the logarithm of 4.57 is 0.6599, and that the logarithm of  $A$  is  $a$ . Then

$$10^{0.6599} = 4.57. \quad 10^a = A.$$

$$\text{But } 10^3 = 1000 \quad 10^3 = 1000.$$

$$\text{Hence } 10^{3.6599} = 4570. \quad 10^{a+3} = 1000 A.$$

It will be noticed that the decimal point of a number is moved to the right, when the number is multiplied by an integral power of 10. Therefore the logarithm of the product will be increased by this integer (the exponent of the multiplier). But adding an integer to a logarithm does not affect the fractional part of the logarithm. Hence the mantissa of the logarithm is unchanged.

In a similar manner, it may be shown that a motion of the decimal point to the left will leave the mantissa of its logarithm unchanged.

To prove the second of these principles consider the following table of logarithms, the base of the system being 10:

| NUMBERS. | LOGARITHMS. |
|----------|-------------|
| .0001    | - 4         |
| .001     | - 3         |
| .01      | - 2         |
| .1       | - 1         |
| 1        | 0           |
| 10       | 1           |
| 100      | 2           |
| 1000     | 3           |
| 10000    | 4           |

From an inspection of the table it will be seen that all numbers between 1000 and 10,000 (numbers with the first significant figure three places to the left of the units place) will have logarithms between 3 and 4, and hence  $3 +$  a proper fraction; and in general that numbers with the first significant figure  $n$  places to the left of the units place will have their logarithms between  $n$  and  $n + 1$ , and hence  $n +$  a

proper fraction. Also that all numbers between .0001 and .001 (numbers with the first significant figure four places to the right of the units place) will have their logarithms between  $-4$  and  $-3$ , and hence  $-4 +$  a proper fraction; and in general that numbers with the first significant figure  $n$  places to the right of the units place will have their logarithms between  $-n$  and  $-(n - 1)$ , and hence  $-n +$  a proper fraction.

For example: The number 4570 is between  $10^3$  and  $10^4$ ; hence its logarithm is  $3 +$  a proper fraction. The number .00457 is between  $10^{-3}$  and  $10^{-2}$ ; hence its logarithm is  $-3 +$  a proper fraction.

It follows, from these two principles, that in the construction of a table of logarithms to the base 10:

(1) The mantissæ need be given but once for numbers having the same figures in the same order.

(2) The characteristics need not be given, but can be supplied by the following rule:

Make the characteristic equal to the number of places by which the first significant figure of the number is removed from the units place: positive if the first significant figure is to the left of the decimal point, negative if to the right.

### THE USE OF THE TABLE OF LOGARITHMS.

**22. To find the logarithm of a number.** — The mantissæ of the logarithms of numbers from 0 to 1000 are printed on pages viii and ix of the tables at the back of the book. If any mantissa is printed without the decimal point, a figure of the mantissa has also been omitted; the decimal point and missing figure will be found above and to the left of the figures in the table.

The first two significant figures of the number will be found in the column marked *N*, and the third significant figure at the top of the columns. If a given number has but one significant figure, two zeros are added; if but two significant figures, one zero is added. Thus:

| PAGE VIII. |         |            | PAGE IX. |         |            |
|------------|---------|------------|----------|---------|------------|
| LINE.      | NUMBER. | LOGARITHM. | LINE.    | NUMBER. | LOGARITHM. |
| 11         | 2.00    | 0.3010     | 11       | .065    | .8129 — 2  |
| 14         | 23.0    | 1.3617     | 14       | .068    | .8325 — 2  |
| 17         | 2.61    | 0.4166     | 17       | .717    | .8555 — 1  |
| 20         | 29.4    | 1.4683     | 20       | .074    | .8692 — 2  |
| 23         | 32.7    | 1.5145     | 23       | .777    | .8904 — 1  |
| 36         | 456     | 2.6590     | 36       | .009    | .9542 — 3  |
| 38         | 4.70    | 0.6721     | 38       | .920    | .9638 — 1  |

To find the logarithm of a number of **four significant figures**, the assumption is made that **small changes in the numbers produce proportional changes in the logarithms**. This assumption is only approximately true; that it is nearly true may be seen from an inspection of the table. The error in the assumption will not, in general, affect the fourth figure of the mantissa of the logarithm.

Thus, let  $m$  stand for mantissa and  $\log$  for logarithm. Then

$$\begin{aligned} m \log 200.2 &= m \log 200 + .2(m \log 201 - m \log 200); \\ &= .3010 + .2(.3032 - .3010). \end{aligned}$$

Therefore  $\log$  of  $200.2 = 2.3014$ .

$$\begin{aligned} m \log .02307 &= m \log 230 + .7(m \log 231 - m \log 230); \\ &= .3617 + .7(.3636 - .3617). \end{aligned}$$

Therefore  $\log$  of  $.02307 = .3630 - 2$ .

**Given a logarithm, to find the corresponding number.** — Let  $\log N = 1.4997$ . Then (page viii, line 22) the significant figures of  $N$  are 316. The characteristic indicates that the figure 3 is one figure to the left of the units place; hence the figure 1 is in the units place and  $N = 31.6$ .

Let  $\log N = .5453 - 2$ . The significant figures of  $N$  are 351, and since the 3 is two figures to the right of the units place,  $N = 0.0351$ .

Let  $\log N = 2.3014$ . Then

$$N = 200 + \frac{.3014 - .3010}{.3032 - .3010} = 200.2.$$

Let  $\log N = .3630 - 2$ . Then

$$N = .0230 + \frac{.3630 - .3617}{.3636 - .3617} = .02307.$$

By assuming either column in this table find the other.

| NUMBER. | LOGARITHM. | NUMBER. | LOGARITHM. |
|---------|------------|---------|------------|
| 10.62   | 1.0261     | 4.426   | 0.6460     |
| 1.384   | 0.1411     | 56.81   | 1.7544     |
| 1886    | 3.2755     | 6.282   | 0.7981     |
| 273.1   | 2.4363     | 733.5   | 2.8654     |

**Other methods of writing negative characteristics.** — To avoid printing negative characteristics, it is usual to arbitrarily increase the characteristic by 10. Thus,  $\log$  of .01657 = .2193 — 2 will sometimes be written 8.2193, which is the logarithm of 165,700,000, or just 10,000,000,000 times as great as the real value. In such a case,  $L$  will be used instead of  $\log$ . Thus

$$\log .01657 = 8.2193 - 10 \text{ and } L. 01657 = 8.2193.$$

Suppose the cube root of .01657 is desired.

Since  $\log .01657 = .2193 - 2$  or  $8.2193 - 10$ ,

$$\log \sqrt[3]{.01657} = .0731 - .6667 \text{ or } 2.7398 - 3.3333.$$

To avoid this negative mantissa, the logarithm of .01657 is arranged before division, thus:

$$\log .01657 = 1.2193 - 3 \text{ or } 28.2193 - 30.$$

$$\text{Therefore } \log \sqrt[3]{.01657} = .4064 - 1 \text{ or } 9.4064 - 10.$$

That is: In taking the root of a proper fraction by logarithms, see that the negative part of the logarithm is divisible by the index of the root.

**Negative numbers.** — The suffix  $n$  placed after a logarithm indicates that the number corresponding to the logarithm is negative. If, in obtaining the product of several numbers, there is an odd number of negative factors, the suffix  $n$  will

occur an odd number of times; therefore, by the principles of algebra, the logarithm of the product should have the suffix  $n$ . But if there is an even number of negative factors, the suffix  $n$  will appear an even number of times; therefore the logarithm of the product will not contain the suffix  $n$ .

In a similar manner, the rules of algebra will serve as a guide in the use of the suffix  $n$ , in the operations of division, involution, and evolution.

## 1. Compute

$$x = 46.12 \times .7827.$$

$$\log 46.12 = 1.6639$$

$$\log .7827 = \underline{9.8936 - 10}$$

$$\log x = \underline{11.5575 - 10}$$

$$x = 36.1$$

## 2. Compute

$$x = 525.1 \div 78530.$$

$$\log 525.1 = 12.7202 - 10$$

$$\log 78530 = \underline{4.8950}$$

$$\log x = \underline{7.8252 - 10}$$

$$x = 0.006687$$

## 3. Compute

$$x = 5850 \div 0.04754.$$

$$\log 5850 = 13.7672 - 10$$

$$\log .04754 = \underline{8.6771 - 10}$$

$$\log x = \underline{5.0901}$$

$$x = 123100$$

## 4. Compute

$$x = \sqrt[3]{\frac{162.4 \times 238.9 \times 115.5}{516.8}}$$

$$\log 162.4 = 2.2106$$

$$\log 238.9 = 2.3782$$

$$\log 115.5 = \underline{2.0626}$$

$$\underline{6.6514}$$

$$\log 516.8 = \underline{2.7133}$$

$$\log x^2 = \underline{3.9381}$$

$$\log x = \underline{1.9690}$$

$$x = 93.12$$

## 5. Compute

$$x = \sqrt[3]{\frac{129.97 \times .0499}{278.7}}$$

$$\log 129.97 = 2.1138$$

$$\log .0499 = \underline{8.6981 - 10}$$

$$\underline{10.8119 - 10}$$

$$\log 278.7 = \underline{2.4451}$$

$$\log x^3 = \underline{28.3668 - 30}$$

$$\log x = \underline{9.4556 - 10}$$

$$x = 0.2855$$

Exercises. — Use problems on page 10.

## III. LOGARITHMS OF FUNCTIONS.

**23.** The table, pages x to xvi, Logarithms of Functions, contains the logarithms of the natural functions of angles from  $0^\circ$  to  $90^\circ$ . To avoid printing negative characteristics, the characteristics of logarithms of numbers less than 1 have been printed 10 too large (indicated by the letter *L* in the first, second, and sixth columns). Hence  $-10$  will be understood after such logarithms.

Just as in the other tables, any number printed without the decimal point will have also one or more figures missing, which figures will be found above and to the left.

The method of interpolation is the same as in the use of the table of natural functions.

**Exercises.**—Copy a value in any row in the following table; then find each of the remaining values in the row:

|                     |                     |                 |                     |                     |
|---------------------|---------------------|-----------------|---------------------|---------------------|
| sin $18^\circ 5'$   | sin $161^\circ 55'$ | 9.4919          | cos $71^\circ 55'$  | cos $288^\circ 5'$  |
| sin $26^\circ 6'$   | sin $153^\circ 54'$ | 9.6434          | cos $63^\circ 54'$  | cos $296^\circ 6'$  |
| sin $25^\circ 52'$  | sin $154^\circ 8'$  | 9.6398          | cos $64^\circ 8'$   | cos $295^\circ 52'$ |
| sin $196^\circ 14'$ | sin $343^\circ 46'$ | 9.4465 <i>n</i> | cos $106^\circ 14'$ | cos $253^\circ 46'$ |
| sin $251^\circ 31'$ | sin $288^\circ 29'$ | 9.9770 <i>n</i> | cos $161^\circ 31'$ | cos $198^\circ 29'$ |
| tan $112^\circ 14'$ | tan $292^\circ 14'$ | 0.3885 <i>n</i> | cot $157^\circ 46'$ | cot $337^\circ 46'$ |
| tan $101^\circ 46'$ | tan $281^\circ 46'$ | 0.6813 <i>n</i> | cot $168^\circ 14'$ | cot $348^\circ 14'$ |
| tan $103^\circ 17'$ | tan $283^\circ 17'$ | 0.6269 <i>n</i> | cot $166^\circ 43'$ | cot $346^\circ 43'$ |
| tan $177^\circ 37'$ | tan $357^\circ 37'$ | 8.6193 <i>n</i> | cot $92^\circ 23'$  | cot $272^\circ 23'$ |
| tan $34^\circ 44'$  | tan $214^\circ 44'$ | 9.8409          | cot $55^\circ 16'$  | cot $235^\circ 16'$ |
| sec $22^\circ 13'$  | sec $337^\circ 47'$ | 0.0335          | csc $67^\circ 47'$  | csc $112^\circ 13'$ |
| sec $142^\circ 23'$ | sec $217^\circ 37'$ | 0.2144 <i>n</i> | csc $232^\circ 23'$ | csc $307^\circ 37'$ |
| sec $76^\circ 8'$   | sec $283^\circ 52'$ | 0.6204          | csc $13^\circ 52'$  | csc $166^\circ 8'$  |
| sec $96^\circ 26'$  | sec $263^\circ 34'$ | 0.9506 <i>n</i> | csc $186^\circ 26'$ | csc $353^\circ 34'$ |
| sec $69^\circ 36'$  | sec $290^\circ 24'$ | 0.4577          | csc $20^\circ 24'$  | csc $159^\circ 36'$ |

## CHAPTER III.

### RIGHT TRIANGLES.

**24.** The primary object of trigonometry is the investigation of the methods of computing the numerical values of the sides and angles of plane triangles. When, from any data, such numerical values have been obtained, the triangle will be said to be **solved**.

With reference to the number of possible answers, a problem is said to be **indeterminate**, **ambiguous**, **determinate**, or **impossible**:

(1) When there is an infinite number of answers, the problem is indeterminate. An algebraic illustration is: find  $x$  and  $y$  from the equation  $x - y = 3$ .

(2) When there is more than one answer, but not an infinite number of answers, the problem is ambiguous. For example: find  $x$  from the equation  $x^2 + 6 = 5x$ .

(3) When there is one answer and but one answer for each unknown, the problem is determinate. For example: find  $x$  and  $y$  from the equations  $x + y = 7$  and  $x - y = 3$ .

(4) When there is no answer, the problem is impossible. For example: find  $x$  from the equation

$$2x - 5 = \text{negative square root of } (x^2 - 7).$$

In trigonometry, **negative and imaginary answers are**, in general, **rejected**, as not indicating true solutions.

It is known from geometry that at least three independent conditions are necessary to make the solution of a

triangle determinate.\* In most cases, three such conditions are sufficient.

Two classes of triangles will be considered, — right and oblique. Any oblique triangle can be divided into two right triangles by letting fall from any vertex a perpendicular upon the opposite side. Hence the solution of all triangles depends upon the solution of right triangles.

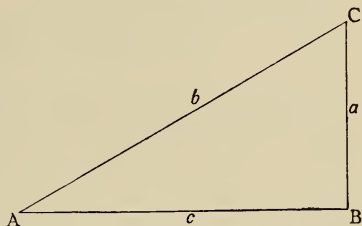


FIG. 27.

**25.** In a right triangle, let  $A$  denote the value of angle  $A$ ,  $B$  denote the right angle, and  $C$  the value of angle  $C$ . Also let  $a$  and  $c$  denote the lengths of the sides of the triangle opposite  $A$  and  $C$ , respectively, and  $b$  denote the length of the hypotenuse.

**Relations of angles.** — In a right triangle, one of the three conditions necessary to determine its size and its shape is explicitly given. For the angle  $B = 90^\circ$ . Therefore

$$A + C = 90^\circ; A = 90^\circ - C; C = 90^\circ - A.$$

Hence, when either acute angle of a right triangle is given, the other acute angle is readily found.

**Relations of sides.** — The relation existing between the sides of a right angle is well known. It is

$$b = \sqrt{c^2 + a^2}; c = \sqrt{(b+a)(b-a)}; a = \sqrt{(b+c)(b-c)}.$$

Hence, if any two of the sides of a right triangle are given, the other side can be computed.

---

\* This is expressed by saying that the triangle is of triple manifoldness.

Compute  $a$ , given  $b = 145$ ,  $c = 143$ .

$$a = \sqrt{(145 + 143)(145 - 143)} = 24.$$

Compute  $c$ , given  $a = 399$ ,  $b = 401$ .

$$c = \sqrt{(401 + 399)(401 - 399)} = 40.$$

**Exercises.** — Given  $b$  and  $c$ ; find  $a$  in each of the following:

| $b$   | $c$    | $a$   |
|-------|--------|-------|
| 919.2 | 918.94 | 21.90 |
| 865.1 | 860.74 | 86.90 |
| 888.9 | 888.84 | 10.34 |
| 150.0 | 149.50 | 12.29 |
| 525.1 | 523.84 | 36.35 |

**Relations of sides and angles.** — Assume  $AB$  to be an initial line,  $c$  the abscissa,  $a$  the ordinate, and  $b$  the distance of  $C$ .

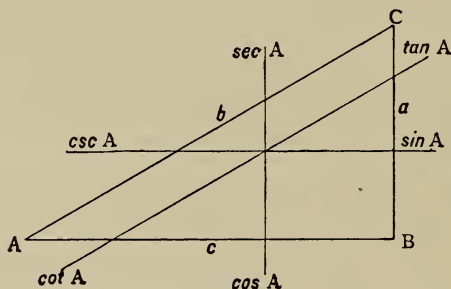


FIG. 28.

The rules on page 19 give

Any required side = product of adjacent side and adjacent function.

Any function of  $A$  = quotient of the adjacent side by the next side.

If it is desired to use the angle  $C$  in a computation, the letters  $a$  and  $c$ ,  $A$  and  $C$ , can be interchanged in Fig. 28, and the rules applied to the resulting figure.

## SOLUTIONS.

**26.** 1. Given  $b = 120$ ,  $A = 11^\circ 25'$ ; find  $a$ .

The horizontal cross-line in Fig. 28 connects the given side  $b$  and the required side  $a$ ; hence by the rule,  $a = b \sin A$ .

$$\begin{aligned}\text{That is,} \quad a &= 120 \sin 11^\circ 25' \\ &= 120 \times .1979 = 23.75.\end{aligned}$$

2. Given  $c = 50.7$ ,  $A = 57^\circ 7'$ ; find  $b$ .

Here we use the vertical cross-line, since that connects the given side and the required side. Hence  $b = c \sec A$ .

$$\begin{aligned}\text{That is,} \quad b &= 50.7 \sec 57^\circ 7' \\ &= 50.7 \times 1.8419 = 93.38.\end{aligned}$$

3. Given  $a = 71$ ,  $A = 65^\circ 32'$ ; find  $c$ .

The oblique cross-line gives

$$\begin{aligned}c &= a \cot A = 71 \cot 65^\circ 32' \\ &= 71 \times .4550 = 32.3.\end{aligned}$$

4. Given  $b = 75$ ,  $a = 43.8$ ; find  $A$ .

Here  $A$  can be found by means of its sine or its cosecant. As the division by 75 is slightly easier, we select the sine.

$$\sin A = \frac{43.8}{75} = .5840; \quad A = 35^\circ 44'.$$

5. Given  $a = 849$ ,  $c = 920$ ; find  $A$ .

Here the oblique cross-line connects the two given sides.  
As 920 is the easier divisor,

$$\tan A = \frac{849}{920} = .9228; A = 42^\circ 42'.$$

6. Given  $b = 401$ ,  $c = 399$ ; find  $A$ .

Here  $\cos A = \frac{399}{401} = .9950.$

In this solution,  $A$  cannot be determined with much precision as the table gives  $5^\circ 43'$ ,  $5^\circ 44'$ , or  $5^\circ 45'$ .

Hence find  $a$  first;  $a = 40$ . Then

$$\csc A = \frac{401}{40} = 10.025; A = 5^\circ 43\frac{1}{2}'.$$

It will be noticed that the cosines and secants of small angles change slowly; and that sines and cosecants of angles near  $90^\circ$  change slowly. When precision is desired, such functions of these angles are avoided.

The following table gives the three sides and one angle of eight different right triangles. Assume any two parts of any one of these triangles and compute another part.

| $a$ | $b$ | $c$ | $A$            |
|-----|-----|-----|----------------|
| 4   | 5   | 3   | $53^\circ 8'$  |
| 12  | 13  | 5   | $67^\circ 23'$ |
| 24  | 25  | 7   | $73^\circ 44'$ |
| 20  | 29  | 21  | $43^\circ 36'$ |
| 12  | 37  | 35  | $18^\circ 55'$ |
| 28  | 53  | 45  | $31^\circ 53'$ |
| 60  | 109 | 91  | $33^\circ 24'$ |
| 84  | 205 | 187 | $24^\circ 11'$ |

## PROBLEMS.

[Answers to these problems will be found on page 177.]

**27. 1.** A tower,  $BC$ , stands on flat horizontal ground; a distance of 250 feet ( $c$ ) was measured from the base of the tower to a point  $A$ . At this point the angle between the horizontal ( $AB$ ) and the visual ray ( $AC$ ) to the top of the tower was found to be  $A = 22^\circ 20'$ .\* From these data, compute the height of the tower.

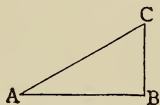


FIG. 29.

**2.** A lighthouse,  $BC$ , is 87 ( $a$ ) feet high. From the top  $C$ , the angle between a horizontal line  $CD$  and the visual ray to a boat  $A$  is found to be  $12^\circ 40'$ .† From these data compute the distance of the boat from the base of the lighthouse.

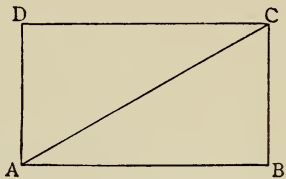


FIG. 30.

**3.** Two forces,  $AB$  of 240 lbs. and  $AD$  of 100 lbs., act at right angles upon the same point of a body. Find the angle which their resultant  $AC$  makes with the larger force.

**4.** The length of the projection of  $AC$  on a line  $AB$  is 23 inches; the angle which  $AC$  makes with this projection is  $38^\circ$ . Find the length of  $AC$ .

**5.** A cannon ball, fired in the air at an angle of  $33^\circ 3'$  with the horizon, passes over 880 feet in one second. How high is it at the end of the second? ‡

\* This angle is called the **angle of elevation** of the top of the tower, as seen from  $A$ .

† This angle is called the **angle of depression** of the boat as seen from  $C$ .

‡ This increase of height per second is called the **vertical component** of the velocity.

6. In order to measure the breadth of a river, a base line  $AB$  was measured along the bank, the point  $B$  being directly opposite a tree  $C$  on the other bank. The base line  $AB$  was 96 (c) feet, and the angle  $BAC$  was found to be  $52^\circ 30'$  ( $A$ ). What is the breadth of the river? \*

7. When the rays of the sun make with the horizontal plane an angle of  $53^\circ 35'$  ( $A$ ),† a vertical pole throws a shadow 11.8 (c) feet long. Find the length of the pole if its lower end is 1 foot ( $d$ ) underground.

8. From a window of a house, the angle of elevation of the top of a church tower on the opposite side of the street was  $56^\circ 36'$  ( $A$ ), and the angle of depression of the base of the tower was  $10^\circ 24'$  ( $A_1$ ). The street being 60 feet (c) wide, find the height of the steeple.

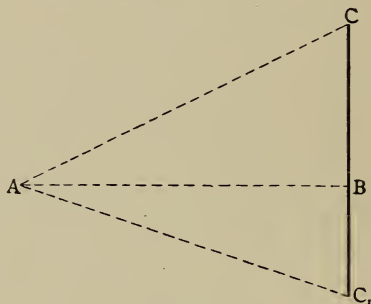


FIG. 31.

9. A house is 24 feet ( $a$ ) high. From the top of the house, the angle of elevation of a church tower on the opposite side of the street is  $36^\circ 54'$  ( $A$ ), and the angle of depression of the base of the tower is  $18^\circ 56'$  ( $A_1$ ). Find the height of the steeple.

10. From the top of a hill, the angles of depression of the top and of the bottom of a house 44 feet ( $a$ ) high are found to be  $28^\circ 4'$  ( $A$ ) and  $43^\circ 36'$  ( $A_1$ ), respectively. Find the height of the hill above the ground on which the house stands.

\* A segment drawn across an angle is said to **subtend** the angle. Thus in a triangle  $ABC$ ,  $BC$  subtends the angle  $A$ .

† The angle which the sun's rays make with the horizon is equal in degrees to the arc which measures the **altitude of the sun**.

**11.** From the top of a bluff overhanging a river, the angle of elevation of the top of a tree on the other side of the river is  $5^{\circ} 22'$  ( $A$ ), and the angle of depression of the bottom of the tree is  $3^{\circ} 40'$  ( $A_1$ ). The tree is known to be 79 feet ( $a$ ) high. What is the breadth of the river?

**12.** A man sailing due north observes two objects directly west. After sailing 12 miles ( $a$ ), the directions of the objects make angles of  $22^{\circ} 38'$  ( $C$ )\* and of  $16^{\circ} 16'$  ( $C_1$ ) with the ship's course. How far apart are the objects?

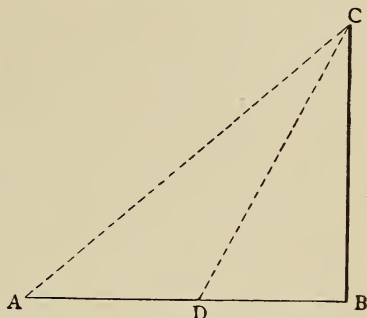


FIG. 32.

**13.** At a horizontal distance of 71 feet ( $c$ ) from the middle of the base of a church tower, the angle of elevation of the top of the tower =  $24^{\circ} 27'$  ( $A$ ), and of the top of the steeple =  $66^{\circ} 10'$  ( $A_1$ ). Find the height of the steeple.

**14.** Two observers, 570 ( $c$ ) feet apart and facing each other, observe a balloon in the vertical plane passing through them, at angles of elevation of  $67^{\circ} 22'$  ( $A$ ) and  $61^{\circ} 56'$  ( $A_1$ ), respectively. What is the height of the balloon above the ground?

**15.** The angle of elevation of a house on the bank of a creek, observed from the opposite bank =  $48^{\circ} 35'$  ( $A$ ). The observer walks straight back from the river a distance of 43.15 feet ( $c$ ), and the angle of elevation is then  $33^{\circ} 24'$  ( $A_1$ ). Find the breadth of the creek.

---

\* The angle which a horizontal line makes with the meridian is called the **bearing** of the line. Thus if  $BC$  is a meridian, and a point  $A$  is  $22^{\circ} 38'$  west of south from a point  $C$ , the bearing of  $A$  from  $C$  is written S.  $22^{\circ} 38'$  W.

16. The angles of elevation of the top of a steeple at distances of 50 feet ( $c$ ) and 450 feet ( $c_1$ ) are found to be complementary. Find the height of the steeple.

17. A man at the top of a lighthouse observes a boat heading straight for the lighthouse; the angle of depression is  $34^\circ 40'$  ( $A$ ); ten minutes ( $t$ ) later, it is  $55^\circ 35'$  ( $A_1$ ). How soon will the boat reach the lighthouse?

18. Rain falls at an angle of  $75^\circ 30'$  ( $A$ ) with the ground. The wind blows at right angles to a wall, 17.5 feet ( $a$ ) high. How far from the wall can a man stand without getting wet, his height being 5.5 feet ( $h$ )?

19. From the end of the diameter of a circle, a chord is drawn, the length of which is  $n$  times the diameter. Find the angle between the chord and the diameter. Take  $n=.62$ .

20. Find the altitude of the sun if the height of a man is  $n$  times the length of his shadow. Take  $n=.74$ .

21. From the top of a lighthouse 221 feet ( $a$ ) above the ocean, two boats are observed, one due west having an angle of depression of  $67^\circ 37'$  ( $A$ ), the other due south having an angle of depression of  $74^\circ 49'$  ( $D$ ). Find the distance between the boats.

22. To measure the height  $BC$  of a tower across a river, the tower being directly opposite a point  $A$ , a distance  $AD = 500$  feet ( $d$ ) was measured along the bank. At  $A$  the angle of elevation of the top of the tower was  $13^\circ 58'$  ( $A$ ), and at  $D$  the elevation was  $10^\circ$  ( $D$ ). Find the height of the tower.

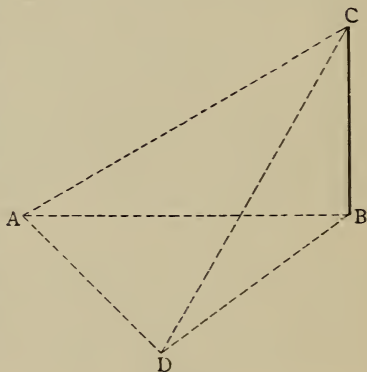


FIG. 33.

THE AREA OF A RIGHT TRIANGLE.

**28.** From the right angle of the triangle, drop a perpendicular upon the hypotenuse. Denote the length of this

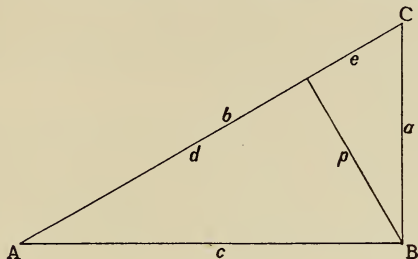


FIG. 34.

perpendicular by  $p$ , and the segments of the hypotenuse by  $d$  and  $e$ . Denote the area by  $K$ .

By geometry,  $K = \frac{1}{2} ac = \frac{1}{2} bp$ .

**Exercises.**—Prove the following relations involving the area of a right triangle:

- |                                               |                                      |
|-----------------------------------------------|--------------------------------------|
| (1) $2K = a^2 \cot A$ .                       | (7) $4K^2 = a^2(b^2 - a^2)$ .        |
| (2) $4K = b^2 \sin 2A$ .                      | (8) $4K^2 = c^2(b^2 - c^2)$ .        |
| (3) $2K = c^2 \tan A$ .                       | (9) $4K = b^2 - (a - c)^2$ .         |
| (4) $K = p^2 \csc 2A$ .                       | (10) $16K^4 - 4a^2d^2K^2 = a^6d^2$ . |
| (5) $4K = (c^2 - a^2) \tan 2A$ .              | (11) $2K = d^2(\tan A + \tan^3 A)$ . |
| (6) $4K^2 = b^3d - b^2d^2$ .                  | (12) $8K = (a + c)^2 - (a - c)^2$ .  |
| (13) $4K = (a + b + c)(a - b + c)$ .          |                                      |
| (14) $8K = [(a + c)^2 + (a - c)^2] \sin 2A$ . |                                      |

If  $K$  is eliminated between any two of these relations, a relation between the parts of the triangle is obtained. Thus if the 2d and 9th relations are used, there is obtained

$$b^2 \sin 2A = b^2 - (a - c)^2.$$

23. A man having traveled 32 miles in a straight line, finds that he has made 17.3 miles more toward the north than toward the east. What direction has he been traveling?

24. A triangular field is at the corner of two roads which form a right angle. The area of the field is 32,130 sq. ft., and the total length of the fence enclosing it is 918 feet. Find the frontage on each road.

25. A triangular field is at the corner of two roads which form a right angle. The area of the field is six-elevenths of an acre, and the length of the fence along the two roads is 487 feet (the total front fence). Find the length of the fence on the back of the field.

## SOLUTIONS USING LOGARITHMS.

29. 1. Given

$$a = 152, A = 18^\circ 25'.$$

Find  $b$  and  $c$ .

$$\log \csc A \quad 0.5004$$

$$\log a \quad 2.1818$$

$$\log \cot A \quad 0.4776$$

$$b = 481.1 \quad \log b \quad 2.6822$$

$$c = 456.5 \quad \log c \quad 2.6594$$

3. Given  $a = .129, c = .089$ .

Find  $A$  and  $b$ .

$$La \quad 9.1106$$

$$Lc \quad 8.9494$$

$$A = 55^\circ 24' \quad \log \tan A \quad 0.1612$$

$$\log \sec A \quad 0.2458$$

$$Lc \quad 8.9494$$

$$b = .1567 \quad Lb \quad 9.1952$$

2. Given

$$c = .3917, A = 65^\circ 14'.$$

Find  $a$  and  $b$ .

$$\log \tan A \quad 0.3360$$

$$Lc \quad 9.5930$$

$$\log \sec A \quad 0.3779$$

$$a = .8492 \quad La \quad 9.9290$$

$$b = .9352 \quad Lb \quad 9.9709$$

4. Given

$$b = 86.53, A = 56^\circ 3'.$$

Find  $a$  and  $c$ .

$$L \sin A \quad 9.9188$$

$$\log b \quad 1.9372$$

$$L \cos A \quad 9.7470$$

$$a = 71.78 \quad \log a \quad 1.8560$$

$$c = 48.33 \quad \log c \quad 1.6842$$

5. Given  $a = 50.7$ ,  $b = 93.4$ . Find the unknown parts and the area. Check the work.

|                              |                |               |
|------------------------------|----------------|---------------|
| $b = 93.4$                   | $\log b$       | <u>1.9703</u> |
| $a = 50.7$                   | $\log a$       | <u>1.7050</u> |
| $A = 32^\circ 53'$           | $\log \csc A$  | <u>0.2653</u> |
| $\therefore C = 57^\circ 7'$ | $\log \cot A$  | <u>0.1894</u> |
|                              | $\log a$       | <u>1.7050</u> |
| $c = 78.42$                  | $\log c$       | <u>1.8944</u> |
|                              | $L .5$         | <u>9.6990</u> |
| $K = 1988$                   | $\log K$       | <u>3.2984</u> |
| $b + a = 144.1$              | $\log (b + a)$ | <u>2.1587</u> |
| $b - a = 42.7$               | $\log (b - a)$ | <u>1.6304</u> |
| Check: $\log c^2$            |                | <u>3.7891</u> |

6. Given  $b = 31.6$ ,  $c = 31.4$ ; find  $a$  and  $A$ .

Here the hypotenuse and side are given nearly equal.  
By (11),

$$\tan \frac{1}{2} A = \csc A - \cot A = \frac{b - c}{a} = \sqrt{\frac{b - c}{b + c}}.$$

|                                                           |                          |               |
|-----------------------------------------------------------|--------------------------|---------------|
|                                                           | $L(b - c)$               | 9.3010        |
|                                                           | $\log (b + c)$           | <u>1.7993</u> |
|                                                           | $\log a^2$               | <u>1.1003</u> |
|                                                           | $L \tan^2 \frac{1}{2} A$ | <u>7.5017</u> |
| $a = 3.549$                                               | $\log a$                 | 0.5501        |
| $\frac{1}{2} A = 3^\circ 13\frac{1}{2}'; A = 6^\circ 27'$ | $L \tan \frac{1}{2} A$   | 8.7508        |

**Exercises.** — Assume any two parts and find the remaining parts in the following right triangles:

1.  $a = 250$     $b = 627$     $c = 575$     $A = 23^\circ 30'$     $K = 71880$
2.  $a = 13.13$     $b = 21.94$     $c = 17.58$     $A = 36^\circ 45'$     $K = 115.4$
3.  $a = 41.54$     $b = 42$     $c = 6.208$     $A = 81^\circ 30'$     $K = 1289$
4.  $a = .3864$     $b = 4.65$     $c = 4.634$     $A = 4^\circ 46'$     $K = .8953$

## PROBLEMS.

30. Solve the following problems, using logarithms:

26. A wall is surrounded by a ditch. A ladder placed with its foot at the edge of the ditch just reaches the top of the wall on the other side of the ditch. The length of the ladder was 33.7 feet, and the angle which it formed with the horizontal plane was  $24^\circ$ . Find the height of the wall and the width of the ditch.

27. Two forces at right angles, one of which is 7.25 pounds, are acting at the same point of a body. Their resultant divides the angle between them into parts proportional to 3 and 5, the smaller angle being next to the given force. Calculate the other component force and the resultant of the two forces.

28. The diagonal of a sheet of paper is 16.5 inches, and it makes, with the shorter side, an angle of  $57^\circ 57'$ . Calculate the sides of the sheet.

29. On flat horizontal ground 154 feet from the foot of a tower, the angle of elevation of the top was  $27^\circ 10'$ . The axis of the telescope was five feet from the ground. Required the height of the tower.

30. The latitude of Philadelphia is  $40^\circ$  N. Taking the radius of the earth at Philadelphia to be 3960 miles, find the radius and the length of  $1^\circ$  of the parallel passing through Philadelphia.

31. The circumference of the meridian passing through Paris is 40,000 kilometers. The length of  $1^\circ$  of the parallel passing through Paris = 73.13 km. Find the latitude of Paris, considering the earth a sphere.

32. Two forces,  $P = 7.25$  lbs., and  $Q = 10.3$  lbs., act at right angles on a body. Calculate the resultant and the angle that it makes with  $P$ .

33. Calculate the velocity per second of a point in  $40^\circ$  latitude as the earth rotates on its axis. Take the radius of the earth = 3960 miles.

34. A frigate is 4 miles west from a steamer which is sailing north at the rate of 22 miles an hour. How far in front of the steamer must the gunner on the frigate aim in order to strike the steamer, if the average velocity of the cannon ball for that distance is 800 feet per second?

35. A square is inscribed in another square; the parts of the sides of the outer square made by the corners of the inner square are 965 ( $m$ ) and 603 ( $n$ ). Find the angle made by the diagonal of the inner square and the side of the outer square.

36. An observer sees a cloud in the southwest at an angle of elevation of  $43^\circ 35'$ . Another observer 2526 feet ( $b$ ) south of the first sees the same cloud in the northwest. How high is the cloud?

37. A pedestal 12 feet high ( $a$ ) supports a column 13 feet high ( $h$ ). An observer, whose eye is in the same horizontal plane with the base of the pedestal, sees the pedestal and column each subtending the same angle. What is the horizontal distance of the observer from the pedestal?

## CHAPTER IV.

### THE ISOSCELES TRIANGLE AND THE REGULAR POLYGON.

**31.** The isosceles triangle can be divided into two equal right triangles by drawing an altitude upon the unequal side. Hence the formulæ for right triangles will apply to the half-triangle  $AB_1C$ . That is,

$$p = b \sin A = \frac{1}{2} c \tan A.$$

$$b = \frac{1}{2} c \sec A = p \csc A.$$

$$c = 2 p \cot A = 2 b \cos A.$$

$$\begin{aligned} \text{Area} = K &= \frac{1}{2} cp = \frac{1}{2} bc \sin A \\ &= \frac{1}{2} b^2 \sin 2A = \frac{1}{4} c^2 \tan A. \end{aligned}$$

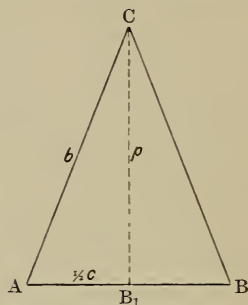


FIG. 35.

In these relations,  $A$  may be changed to  $\frac{1}{2} C$  and the co-functions taken.

**Exercises.** — Assume any two parts in any row of the following table and compute the remaining parts in that row:

| $p$   | $b$   | $c$   | $A$            | $K$   |
|-------|-------|-------|----------------|-------|
| 16.80 | 17.00 | 5.200 | $81^\circ 12'$ | 43.68 |
| 1.960 | 2.275 | 2.310 | $59^\circ 29'$ | 2.264 |
| 70.00 | 101.5 | 147.0 | $43^\circ 36'$ | 5145  |
| 10.80 | 33.30 | 63.00 | $18^\circ 55'$ | 340.2 |
| 3.179 | 4.264 | 5.684 | $48^\circ 12'$ | 9.034 |
| 25.00 | 62.70 | 115.0 | $23^\circ 30'$ | 1438  |
| .5878 | 1.000 | .1618 | $36^\circ 00'$ | .4755 |
| 18.08 | 83.50 | 163.0 | $12^\circ 30'$ | 1474  |

## PROBLEMS.

**32. 38.** Two trains start at the same station at the same time on tracks making an angle of  $67^{\circ} 30'$ . Each goes with a velocity of 27 miles per hour. In how many minutes will the distance between them be equal to 5 miles?

**39.** The chord of a circle is 27 inches, and the angle at the center subtended by the chord is  $116^{\circ} 12'$ . Find the radius.

**40.** The angle at the vertex of an isosceles triangle is  $87^{\circ} 13'$ , and the sum of the two unequal altitudes is 50. Find the base.

**41.** The ratio of the altitude from the vertex of an isosceles triangle to one of the equal altitudes is 1.2, and each of the equal sides = 5.3. Find the base angle and the area.

**42.** From a point without a circle 12 feet ( $d$ ) from the center, two tangents are drawn, making with each other an angle of  $25^{\circ} 3'$  ( $A$ ). Calculate the area of an equilateral triangle inscribed in this circle.

**43.** From a point without a circle 23.2 feet from the center, two tangents are drawn. The radius of the circle equals 9.6 feet. Calculate the side of an equilateral triangle whose area is equal to that of the figure bounded by the two tangents and by the smaller arc included between the points of contact.

**44.** The slant height of a right cone is 13 feet, and it makes with the plane of the base an angle of  $67^{\circ} 23'$ . Find the convex surface of the cone.

**45.** When a cone was cut through its axis, the area of the section was found to be 10 sq. inches and the angle at the vertex  $40^{\circ} 30'$ . Find the convex surface of the cone.

46. The convex surface of a cone = 80 sq. inches ( $S$ ), and the angle at the vertex of the principal section =  $39^{\circ} 13'$  ( $C$ ). Find the volume of the cone.

47. When the convex surface of a right cone was spread out in a plane, it was found to be a sector of a circle whose central angle was  $108^{\circ} 36'$ , and the chord subtending the corresponding arc was found to be 12 inches. Calculate the volume of the cone.

48. The altitude to the base of an isosceles triangle is 5.4 feet, and one of the equal altitudes is 7.12 feet. Find the area.

49. The area of an isosceles triangle is equal to the area of the semicircle on the base as a diameter. Find the angle at the base.

50. In a circle whose radius is 9.4 inches, a chord of 14.4 inches is drawn. Find the central angle corresponding to this chord.

51. Calculate the area of a circle in which a chord 5 feet long subtends a central angle of  $77^{\circ} 36'$ .

52. The lengths of the sides of a rectangle are 78 ft. and 71 ft. The middle points of its sides are the corners of an inscribed rhombus. Find an acute angle of this rhombus.

53. A point moves in the arc of a semicircle of radius  $b$  with a uniform velocity of  $a$  feet per second. In how many seconds is it at a distance of  $c$  feet (measured on the chord) from its starting point? Take  $b = 10$ ,  $a = 3$ ,  $c = 16$ .

54. Two equal forces act on a body, the angle between them being  $146^{\circ} 48'$ . Their resultant is 12 pounds. Find the component forces.

55. What must be the angle between two equal forces, each 3.5 pounds, if their resultant is 1 pound?

## THE REGULAR POLYGON.

**33.** Denote the number of sides by  $n$ . Then the half angle at the center is  $\frac{180^\circ}{n}$ , which denote by  $A$ . Also denote the side by  $a$ , the radius of the circumcircle by  $R$ , and the radius of the incircle by  $r$ . Then

$$a = 2 R \sin A = 2 r \tan A.$$

$$R = r \sec A = \frac{1}{2} a \csc A.$$

$$r = \frac{1}{2} a \cot A = R \cos A.$$

$$\text{Area polygon} = \frac{1}{2} n a r = \frac{1}{4} n a^2 \cot A.$$

$$\text{Area incircle} = \pi r^2 = \frac{1}{4} \pi a^2 \cot^2 A.$$

$$\text{Area circumcircle} = \pi R^2 = \frac{1}{4} \pi a^2 \csc^2 A.$$

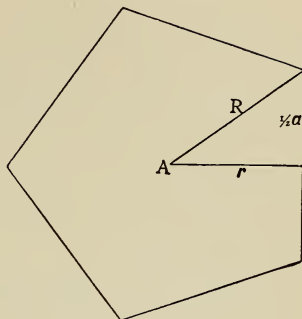


FIG. 36.

Prove the following properties of regular polygons:

The area of the ring between the incircle and circumcircle of a regular polygon is equal to the area of the circle on one side as a diameter.

The perimeter of a regular polygon : the perimeter of its incircle = area of polygon : area of incircle.

The area of a regular inscribed polygon of an even number of sides is a mean proportional between the areas of the regular inscribed polygon and the regular circumscribed polygon of half the number of sides.

**Stars.** — Divide the circumference of a circle into  $n$  equal parts. Let  $s$  denote a whole number less than  $\frac{1}{2}n$ ; join every  $s$ th point of division of the circumference (or every  $n - s$ th point). There is obtained an  $n$ -pointed star of species  $s$ .

1. If  $s$  is 1, the perimeter is continuous and non-intersecting; the figure consists of a regular polygon of  $n$  sides.

2. If  $n$  is a multiple of  $s$ , say  $n = rs$ , the perimeter is discontinuous; the figure consists of  $s$  regular polygons each of  $r$  sides. For example, the fifteen-pointed star of species 3 consists of 3 regular pentagons.

3. If  $n$  and  $s$  have a common divisor, say  $n = ru$  and  $s = rv$ , the perimeter is discontinuous; the figure consists of  $r$  different stars each of them having  $u$  corners. For example, the fifteen-pointed star of species 6 consists of 3 five-pointed stars.

4. If  $n$  and  $s$  are prime to each other, the perimeter is continuous and intersecting. For example, the fifteen-pointed star of species 4.

**Properties.** — 1. In a star, prove that the sum of the  $n$  angles  $= (n - 2s) 180^\circ$ .

2. Prove that  $a = 2R \sin \frac{s}{n} 180^\circ = 2r \tan \frac{s}{n} 180^\circ$ ,

$R$  being the radius of the circumcircle,  $r$  the radius of the incircle, and  $a$  the length between any two points joined in the construction.

**Exercises.** — The following table gives elements of seven regular polygons. Assume the number of sides and any one element to compute the other elements.

| $n$ | $a$   | $R$   | $r$   | $K$  |
|-----|-------|-------|-------|------|
| 5   | 24.68 | 21.00 | 16.99 | 1049 |
| 6   | 20.78 | 20.78 | 18.00 | 1122 |
| 7   | 13.88 | 16.00 | 14.41 | 700  |
| 8   | 33.14 | 43.30 | 40.00 | 5300 |
| 12  | 14.00 | 27.00 | 26.12 | 2195 |
| 20  | 6.257 | 20.00 | 19.75 | 1236 |
| 25  | 10.90 | 43.50 | 43.16 | 5880 |

## PROBLEMS.

**34. 56.** The diagonal of a regular pentagon is 4.3 feet ( $d$ ). Find the area of the pentagon.

**57.** The area of a circle is 15 sq. feet. Find the area of a regular heptagon whose perimeter is equal to the circumference of the given circle.

**58.** Find the largest and smallest diagonals of a regular polygon of  $7(2n + 1)$  sides, each side being 4 yards ( $a$ ).

**59.** In a park it is desired to make a flower garden to contain one acre, and to be of the form of a regular five-pointed star. Find each side of the bounding decagon.

**60.** Regular polygons of 27 sides are inscribed in and circumscribed about a circle whose radius is 9.2, the sides of the two polygons being respectively parallel. What is the area of the figure included between the perimeters of the two polygons?

**61.** The area of a regular inscribed polygon of  $n$  sides is  $K$ . Find the area of a regular polygon of  $m$  sides circumscribed about the same circle. Put  $n = 9, m = 7, K = 100$ .

**62.** In a circle whose radius is 2 feet two parallel chords are drawn on the same side of the center. One is equal to the side of the regular inscribed polygon of 9 sides, the other to the side of the regular circumscribed hexagon (about the same circle). Calculate the area of the part included between these two chords.

**63.** The radius of a circle is 7 yards. A chord is drawn equal to the side of the regular inscribed polygon of 15 sides. In the larger of the two segments thus formed, a circle is inscribed touching the arc in its middle point. Calculate the area of a regular polygon of 9 sides circumscribed about the latter circle.

## CHAPTER V.

### OBLIQUE TRIANGLES.

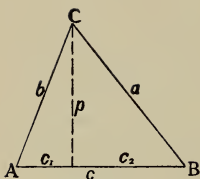


FIG. 37.

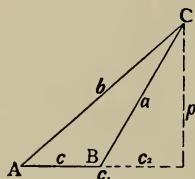


FIG. 38.

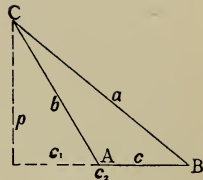


FIG. 39.

**35. Notation.** — In an oblique triangle, let  $A$  denote the value of angle  $A$ ,  $B$  the value of angle  $B$ ,  $C$  the value of angle  $C$ ;  $a$  the length of the side opposite  $A$ ,  $b$  the length of the side opposite  $B$ ,  $c$  the length of the side opposite  $C$ ;  $p$  the perpendicular from  $C$  on  $c$ ,  $c_1$  the distance from  $A$  to the foot of the perpendicular, and  $c_2$  the distance from  $B$  to the foot of the perpendicular. The length  $c_1$  will be regarded as positive when drawn from  $A$  toward  $B$ , and negative when drawn in the opposite direction;  $c_2$  will be regarded as positive when drawn from  $B$  toward  $A$ , and negative when drawn in the opposite direction.

**Relation between the angles.** — From geometry,

$$A + B + C = 180^\circ.$$

Hence, when any two angles of a triangle are given, the third angle can readily be found.

**36. Methods of Solution.** — At least three independent conditions must be given to make the solution of an oblique triangle determinate. Suppose that these three conditions are the values of sides or angles. When possible, draw an altitude such that two of the given parts are in one triangle.

Now find a value of this altitude in terms of two known parts in the one right triangle. Also in the other right triangle find a value of this same altitude in terms of the other given part and a desired part. These values of the altitude give an equation in which the desired part is the only unknown. Thus,

**Given one side and two angles:** as  $b$ ,  $A$  and  $B$ . To find  $a$ ,  
equate values of  $p$  thus:  $p = a \sin B = b \sin A$ .

Multiply by  $\csc B$ :  $a = b \csc B \sin A$ .

If  $C$  and  $c$  are also desired:  $C = 180^\circ - (A + B)$ .

By drawing the altitude from  $A$  on  $a$ ,

$$c \sin B = b \sin C, \text{ whence } c = b \csc B \sin C.$$

These values of  $a$  and  $c$  give the following relation:

$$a \csc A = b \csc B = c \csc C. \quad (19)$$

**Given two sides and the angle opposite one of these sides:**  
as  $a$ ,  $b$ , and  $A$ . Equate values of  $p$ :  $a \sin B = b \sin A$ ;

whence  $\sin B = \frac{b}{a} \sin A,$

$C$  and  $c$  being found as in the last case.

**Given two sides and the included angle:** as  $b$ ,  $c$ , and  $A$ .

First find  $c_1 = b \cos A$  and  $c_2 = c - c_1$ .

To find  $B$ , equate values of  $p$  thus:

$$p = c_2 \tan B = c_1 \tan A; \text{ whence } \tan B = \frac{c_1}{c_2} \tan A.$$

To find  $a$ , equate values of  $p^2$ :

$$a^2 - c_2^2 = b^2 - c_1^2. \quad \text{Put } c - c_1 \text{ for } c_2.$$

$$a^2 = b^2 + (c - c_1)^2 - c_1^2 = b^2 + c^2 - 2cc_1.$$

$$\text{Put } b \cos A \text{ for } c_1; \text{ then } a^2 = b^2 + c^2 - 2bc \cos A. \quad (20)$$

**Given the three sides:** as  $a$ ,  $b$ , and  $c$ .

$$\text{From (20),} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad (21)$$

with similar relations for the other angles.

It is usually inconvenient to compute  $\cos A$  from (21); another method will now be found for computing the values of the angles of a triangle when the three sides are given.

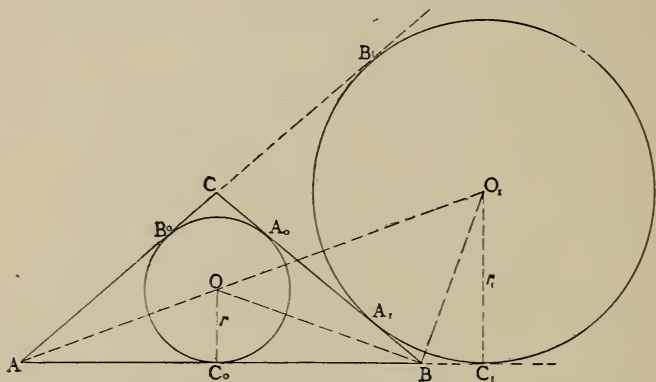


FIG. 40.

Prolong the sides  $AB$  and  $AC$  of triangle  $ABC$ . Bisect angles  $A$ ,  $B$ , and  $CBC_1$ , these bisectors meeting at  $O$  and  $O_1$ . The point  $O$  is the center of the incircle which touches the sides of the triangle in  $A_0$ ,  $B_0$ , and  $C_0$ ; the point  $O_1$  is the center of the excircle in angle  $A$  which touches  $BC$  at  $A_1$ ,  $AC$  prolonged at  $B_1$ , and  $AB$  prolonged at  $C_1$ .

From geometry, the tangents from a point outside of a circle are equal to each other; thus  $AB_0 = AC_0$ ,  $AB_1 = AC_1$ , and so on.

Denote each of the tangents from  $A$  to the incircle by  $x$ , each of the tangents from  $B$  by  $y$ , each of the tangents from  $C$  by  $z$ , and half of the sum of the sides,  $\frac{1}{2}(a + b + c)$  by  $s$ ; then

Denote each of the tangents from  $A$  to the excircle by  $x_1$ , each of the tangents from  $B$  by  $y_1$ , each of the tangents from  $C$  by  $z_1$ , and half of the sum of the sides  $\frac{1}{2}(a + b + c)$  by  $s$ ; then

$$y + z = a,$$

$$x + z = b,$$

$$x + y = c,$$

which give

$$x = AB_0 = AC_0 = s - a,$$

$$y = BC_0 = BA_0 = s - b,$$

$$z = CA_0 = CB_0 = s - c.$$

$$y_1 + z_1 = a,$$

$$x_1 - z_1 = b,$$

$$x_1 - y_1 = c,$$

which give

$$x_1 = AB_1 = AC_1 = s,$$

$$y_1 = BC_1 = BA_1 = s - c,$$

$$z_1 = CA_1 = CB_1 = s - b.$$

Now the two triangles  $AOC_0$  and  $AO_1C_1$  are similar because their sides are respectively parallel; and the two triangles  $BOC_0$  and  $BO_1C_1$  are similar because their sides are respectively perpendicular; hence

$$\frac{OC_0}{O_1C_1} = \frac{AC_0}{AC_1} \quad \text{or} \quad \frac{r}{r_1} = \frac{s-a}{s},$$

$$\frac{OC_0}{BC_1} = \frac{BC_0}{O_1C_1} \quad \text{or} \quad \frac{r}{s-c} = \frac{s-b}{r_1};$$

$r$  and  $r_1$  being the radii of the incircle and excircle respectively.

Multiply these equations and solve for  $r$ .

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\left. \begin{array}{l} \text{In triangle } AOC_0, \quad \cot \frac{1}{2} A = \frac{s-a}{r} \\ \text{Changing letters,} \quad \cot \frac{1}{2} B = \frac{s-b}{r} \\ \quad \quad \quad \cot \frac{1}{2} C = \frac{s-c}{r} \end{array} \right\} \quad (22)$$

The equations in (22) furnish the solution for the angles of a triangle when the sides are given. This method is more suitable for logarithmic computation than the use of (21).

## SOLUTIONS.

One side and two angles.

**37.** 1. Given  $a = 200$ ,  $A = 74^\circ 36'$ ,  $B = 81^\circ 12'$ ; find  $b$ .  
By (19),  $b = a \csc A \sin B = 200 \times 1.0372 \times .9882 = 205$ .

2. With the same data, find  $C$  and  $c$ .

$$C = 180^\circ - (A + B) = 180^\circ - 155^\circ 48' = 24^\circ 12'.$$

By (19),  $c = a \csc A \sin C = 200 \times 1.0372 \times .4099 = 85$ .

3. Given  $b = 37$ ,  $B = 53^\circ 8'$ ,  $A = 107^\circ 57'$ ; find  $a$ .

By (19),  $a = b \csc B \sin A = 37 \times 1.2500 \times .9513 = 44$ .

Two sides and the angle opposite one of them.

1. Given  $A = 30^\circ$ ,  $a = 1$ ,  $b = 4$ ; find  $B$  and  $C$ .

Here  $\sin B = \frac{4}{1} \sin 30^\circ = 4 \times .5 = 2$ .

Since  $\sin B > 1$ , the solution is impossible.

2. Given  $A = 30^\circ$ ,  $a = 2$ ,  $b = 4$ ; find  $B$  and  $C$ .

Here  $\sin B = \frac{4}{2} \sin 30^\circ = 1$ .

Whence  $B = 90^\circ$  and  $C = 180^\circ - (A + B) = 60^\circ$ .

3. Given  $A = 30^\circ$ ,  $a = 3$ ,  $b = 4$ ; find  $B$  and  $C$ .

Here  $\sin B = \frac{4}{3} \sin 30^\circ = .6667$ .

$B = 41^\circ 49'$  or  $138^\circ 11'$ , and  $C = 108^\circ 11'$  or  $11^\circ 49'$ .

Note that the values of  $C$  are obtainable by subtracting  $A$  from the values of  $B$ . Explain this geometrically.

4. Given  $A = 30^\circ$ ,  $a = 4$ ,  $b = 4$ ; find  $B$  and  $C$ .

Here  $\sin B = \frac{4}{4} \sin 30^\circ = .5$ .

$B = 30^\circ$  or  $150^\circ$ , and  $C = 120^\circ$  or  $0^\circ$ .

5. Given  $A = 30^\circ$ ,  $a = 5$ ,  $b = 4$ ; find  $B$  and  $C$ .

Here  $\sin B = \frac{4}{5} \sin 30^\circ = .4$ .

$B = 23^\circ 35'$  or  $156^\circ 25'$ , and  $C = 126^\circ 25'$  or  $-6^\circ 25'$ .

These five examples are illustrated in the geometric construction, Fig. 41. The base is drawn indefinitely, angle  $A$  is made equal to  $30^\circ$ ,  $AC = b = 4$ . Successive radii,  $a_1, a_2, a_3, a_4, a_5$  are used to draw arcs with the center at  $C$ .

In order that the triangles may have the angle  $A = 30^\circ$ , the vertex  $B$  of each of them must be on the half-line  $AB_5$  drawn through  $A$  to the right. Also in order that the triangles may have the side  $a$  of the length given in each problem, the vertex  $B$  must be in the arc appropriate to that problem.

To prevent confusion in the diagram, the side  $a$  has not been drawn, except for the negative solution of question 5. It is recommended that the boundary of the triangles be drawn with crayons of different colors.

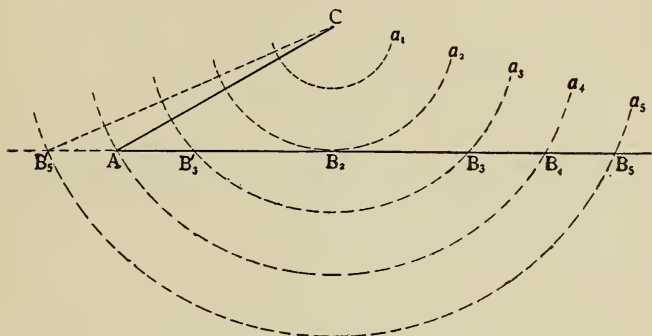


FIG. 41.

It will be noticed that the first arc does not intersect the half-line  $AB_5$ . The second arc touches it at one point. The third arc cuts it at two points, both to the right of  $A$ . The fourth arc cuts it once to the right of  $A$  and once at  $A$ . The fifth arc cuts the half-line  $AB_5$  to the right of  $A$  and also cuts the half-line  $AB'_5$  to the left of  $A$ ; the negative value of  $C = -6^\circ 25'$ , being the angle  $ACB'_5$ .

Solve the following five examples, and compare the results with the geometric construction, Fig. 42.

1. Given  $A = 150^\circ$ ,  $a = 1$ ,  $b = 4$ ; find  $B$  and  $C$ .
2. Given  $A = 150^\circ$ ,  $a = 2$ ,  $b = 4$ ; find  $B$  and  $C$ .
3. Given  $A = 150^\circ$ ,  $a = 3$ ,  $b = 4$ ; find  $B$  and  $C$ .
4. Given  $A = 150^\circ$ ,  $a = 4$ ,  $b = 4$ ; find  $B$  and  $C$ .
5. Given  $A = 150^\circ$ ,  $a = 5$ ,  $b = 4$ ; find  $B$  and  $C$ .

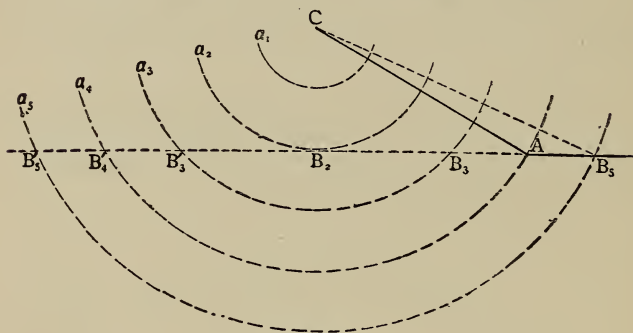


FIG. 42.

Assume a value in each of three columns in any row in the following table, and compute the remaining value (or values) in that row:

| $a$ | $b$ | $A$            | $B$                               |
|-----|-----|----------------|-----------------------------------|
| 14  | 15  | $59^\circ 29'$ | $67^\circ 23'$ or $112^\circ 37'$ |
| 37  | 40  | $67^\circ 23'$ | $93^\circ 42'$ or $86^\circ 18'$  |
| 75  | 77  | $61^\circ 56'$ | $64^\circ 57'$ or $115^\circ 3'$  |
| 69  | 139 | $21^\circ 14'$ | $46^\circ 52'$ or $133^\circ 8'$  |
| 80  | 401 | $11^\circ 27'$ | $84^\circ 17'$ or $95^\circ 43'$  |
| 101 | 120 | $43^\circ 36'$ | $55^\circ 1'$ or $124^\circ 59'$  |
| 148 | 153 | $53^\circ 8'$  | $55^\circ 48'$ or $124^\circ 12'$ |
| 240 | 409 | $34^\circ 7'$  | $72^\circ 56'$ or $107^\circ 4'$  |
| 449 | 560 | $51^\circ 25'$ | $77^\circ 10'$ or $102^\circ 50'$ |

6. Given  $a = 7$ ,  $b = 8$ ,  $A = 60^\circ$ ; find  $c$  directly.

From (20),  $c^2 - 2bc \cos A = a^2 - b^2$ .

Substituting the given numerical values,

$$c^2 - 8c = -15,$$

which gives  $c = 3$  or  $5$ .

7. Given  $a = 7$ ,  $b = 5$ ,  $A = 120^\circ$ ; find  $c$ .

Here the equation is  $c^2 + 5c = 24$ .

Therefore  $c = 3$  or  $-8$ .

**Two sides and the included angle.**

1. Given  $b = 85$ ,  $c = 200$ ,  $A = 81^\circ 12'$ ; find  $B$ .

$$c_1 = b \cos A = 85 \cos 81^\circ 12' = 13.$$

$$c_2 = 200 - 13 = 187.$$

$$p = c_2 \tan B = c_1 \tan A.$$

$$\tan B = \frac{c_1}{c_2} \tan A = .4491$$

Therefore  $B = 24^\circ 11$ .

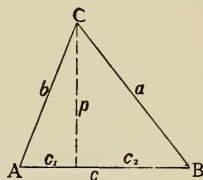


FIG. 43.

2. With the same data, find  $a$ .

$$\begin{aligned} \text{From (20),} \quad a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 7225 + 40000 - 5200 = 42025. \end{aligned}$$

Therefore  $a = 205$ .

3. Given  $b = 37$ ,  $c = 15$ ,  $A = 107^\circ 57'$ ; find  $B$ .

Here  $c_1 = b \cos A = -11.4$ .

$$c_2 = 15 - (-11.4) = 26.4.$$

$$\begin{aligned} \tan B &= \frac{c_1}{c_2} \tan A = -\frac{11.4}{26.4} \times (-3.0868) \\ &= 1.3329. \end{aligned}$$

Therefore,  $B = 53^\circ 7'$ .

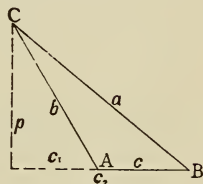


FIG. 44.

4. With the same data, find  $a$ .

$$a^2 = b^2 + c^2 - 2bc \cos A = 1369 + 225 + 342 = 1936.$$

Therefore,  $a = 44$ .

The three sides.

1. Given  $a = 7$ ,  $b = 8$ ,  $c = 5$ ; find  $A$ .

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 25 - 49}{2 \times 8 \times 5} = .5; \quad A = 60^\circ.$$

Or the angle may be found by using the first and second equations in (22); thus,

$$\cot \frac{1}{2} A = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \sqrt{\frac{10 \times 3}{2 \times 5}} = 1.7321; \quad \frac{1}{2} A = 30^\circ.$$

2. Given  $a = 7$ ,  $b = 5$ ,  $c = 3$ ; find  $A$ .

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = -.5. \quad \text{Therefore } A = 120^\circ.$$

$$\cot \frac{1}{2} A = \sqrt{\frac{7.5 \times .5}{2.5 \times 4.5}} = .5774; \quad \frac{1}{2} A = 60^\circ.$$

**Exercises.** — Assume any three parts in each of the following five triangles. Then compute the fourth part:

|                    |          |          |                  |
|--------------------|----------|----------|------------------|
| $A = 60^\circ$     | $a = 13$ | $b = 15$ | $c = 7$ or $8$   |
| $A = 120^\circ$    | $a = 13$ | $b = 7$  | $c = 8$ or $-15$ |
| $A = 53^\circ 8'$  | $a = 13$ | $b = 15$ | $c = 4$ or $14$  |
| $A = 61^\circ 56'$ | $a = 65$ | $b = 68$ | $c = 7$ or $57$  |
| $A = 67^\circ 23'$ | $a = 25$ | $b = 26$ | $c = 3$ or $17$  |

**38.** Given two sides and the included angle, to find a function of another angle directly in terms of the given parts.

Since  $c = c_1 + c_2$ :

$$c = p(\cot A + \cot B).$$

Multiply by  $\csc A$ , and put  $b$  for  $p \csc A$ :

$$c \csc A = b(\cot A + \cot B). \quad (23)$$

$$\text{Whence} \quad \cot B = c \frac{\csc A}{b} - \cot A. \quad (24)$$

After  $B$  has been found,  $a = \csc B \div \frac{\csc A}{b}$ .

The objection to the use of (24) is that in general both tables of functions are used; the table of logarithms of functions in the computation of the value of  $c \frac{\csc A}{b}$ , and natural values of functions in completing the work. In general, this computation is not troublesome; and where results are to be carried through a series of triangles, as in a survey, it is of advantage to be able to write down the function of a desired angle directly in terms of the data.

As a simple application of (23), consider the following problem:

When a point ( $X$ ) in a triangle is joined to the corners, it is found that the angles  $XBA$ ,  $XCB$ , and  $XAC$  are all equal. Find the value of one of these equal angles ( $x$ ).

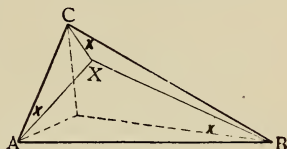


FIG. 45.

In triangle  $AXC$ , by (23);  $AX \csc x = b (\cot x + \cot CXA)$ .

In triangle  $AXB$ , by (19);  $AX \csc x = c \csc AXB$ .

Now angle  $CXA = 180^\circ - (C - x) - x = 180^\circ - C$ .

Similarly,  $AXB = 180^\circ - A$ .

Hence by substituting:  $b (\cot x - \cot C) = c \csc A$ .

But by (23),  $c \csc A = b (\cot A + \cot B)$ . Hence

$$\cot x = \cot A + \cot B + \cot C.$$

If the angles are  $65^\circ$ ,  $30^\circ$ , and  $85^\circ$ , respectively,  $x = 23^\circ 38'$ .

The point  $X$  and another point found by counting the angles  $x$  from the other sides of  $a$ ,  $b$ , and  $c$  are called **Brocard's Points**. They possess many interesting properties; for example, they are the foci of the ellipse inscribed in the triangle.

## THE AREA OF A TRIANGLE.

**39.** Denote the area by  $K$ . Then

By geometry,  $K = \frac{1}{2} cp.$

Since  $p = b \sin A$ ,  $K = \frac{1}{2} bc \sin A.$

Changing letters,  $K = \frac{1}{2} ca \sin B.$

$$K = \frac{1}{2} ab \sin C.$$

$b = a \csc A \sin B$ ,  $K = \frac{1}{2} a^2 \csc A \sin B \sin C.$

Changing letters,  $K = \frac{1}{2} b^2 \sin A \csc B \sin C.$

$$K = \frac{1}{2} c^2 \sin A \sin B \csc C.$$

From Fig. 40,  $K = \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr = rs.$

Substitute from (22),  $K = \sqrt{s(s-a)(s-b)(s-c)}.$

**Prove the following :**  $K = \frac{1}{2} \sqrt{a^2 b^2 c^2 \sin A \sin B \sin C}.$

$$K = r^2 \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C.$$

$$K = r_1^2 \cot \frac{1}{2} A \tan \frac{1}{2} B \tan \frac{1}{2} C.$$

$$K = rr_1 \cot \frac{1}{2} A.$$

$$K = s^2 \tan \frac{1}{2} A \tan \frac{1}{2} B \tan \frac{1}{2} C.$$

$$K = (s-a)^2 \tan \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C.$$

By equating any two values of the area, a relation between segments and angles of the triangle is obtained.

**Exercises.** — Without using logarithms, find the area of each of the following triangles :

$$a = 115 \quad B = 67^\circ 23' \quad C = 46^\circ 24'$$

$$a = 240 \quad b = 53 \quad B = 8^\circ 10'$$

$$a = 200 \quad b = 205 \quad C = 24^\circ 11'$$

$$a = 116 \quad b = 105 \quad c = 143$$

## LOGARITHMIC SOLUTIONS.

40. Given one side and two angles. —

$$a = 24.31, \quad A = 45^\circ 18', \quad B = 22^\circ 11'.$$

Solve the triangle.

$$= a \csc A \sin B, \quad c = a \csc A \sin C.$$

|                        |               |        |
|------------------------|---------------|--------|
|                        | $L \sin B$    | 9.5770 |
|                        | $\log a$      | 1.3858 |
| $A + B = 67^\circ 29'$ | $\log \csc A$ | 0.1483 |
| $C = 112^\circ 31'$    | $L \sin C$    | 9.9656 |
| $b = 12.92$            | $\log b$      | 1.1111 |
| $c = 31.6$             | $\log c$      | 1.4997 |

Given two sides and the angle opposite one of them. —

$$a = 215.9, \quad b = 307.7, \quad A = 25^\circ 10'.$$

Solve the triangle.

$$\sin B = \frac{b}{a} \sin A, \quad C_1 = B_2 - A, \quad C_2 = B_1 - A, \quad c = a \csc A \sin C.$$

|                       |               |        |
|-----------------------|---------------|--------|
|                       | $\log b$      | 2.4881 |
|                       | $\log a$      | 2.3343 |
|                       |               | 0.1538 |
|                       | $L \sin A$    | 9.6286 |
| $B_1 = 37^\circ 18'$  | $L \sin B$    | 9.7824 |
| $B_2 = 142^\circ 42'$ | $L \sin C_1$  | 9.9478 |
| $A = 25^\circ 10'$    | $\log a$      | 2.3343 |
| $C_1 = 117^\circ 32'$ | $\log \csc A$ | 0.3714 |
| $C_2 = 12^\circ 8'$   | $L \sin C_2$  | 9.3226 |
| $c_1 = 450.3$         | $\log c_1$    | 2.6535 |
| $c_2 = 109.2$         | $\log c_2$    | 2.0383 |

Given two sides and the included angle.—

$$b = 767, c = 947, A = 10^\circ 50'.$$

Find the remaining parts.

$$c_1 = b \cos A, \quad c_2 = c - c_1,$$

$$p = c_2 \tan B = c_1 \tan A;$$

$$\text{whence } \tan B = \frac{c_1}{c_2} \tan A, \quad a = c_2 \sec B.$$

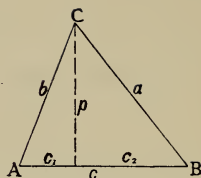


FIG. 46.

|                     |               |        |
|---------------------|---------------|--------|
|                     | $\log b$      | 2.8848 |
| $c = 947$           | $L \cos A$    | 9.9922 |
| $c_1 = 753.3$       | $\log c_1$    | 2.8770 |
| $c_2 = 193.7$       | $\log c_2$    | 2.2871 |
|                     |               | 0.5899 |
| $A = 10^\circ 50'$  | $L \tan A$    | 9.2819 |
| $B = 36^\circ 40'$  | $L \tan B$    | 9.8718 |
| $47^\circ 30'$      | $\log \sec B$ | 0.0958 |
| $C = 132^\circ 30'$ | $\log c_2$    | 2.2871 |
| $a = 241.5$         | $\log a$      | 2.3829 |

Given  $b = 767, c = 947, A = 10^\circ 50'$ ; find  $a$ .

$$\text{By (20),} \quad a = \sqrt{b^2 + c^2 - 2bc \cos A}.$$

|                        |                   |        |
|------------------------|-------------------|--------|
|                        | $\log 2$          | 0.3010 |
|                        | $\log b$          | 2.8848 |
|                        | $\log c$          | 2.9763 |
| $b^2 = 588300$         | $L \cos A$        | 9.9922 |
| $c^2 = 896600$         | $\log b^2$        | 5.7696 |
| 1484900                | $\log c^2$        | 5.9526 |
| $2bc \cos A = 1427000$ | $\log 2bc \cos A$ | 6.1543 |
| $a^2 = 57900$          | $\log a^2$        | 4.7627 |
| $a = 240.6$            | $\log a$          | 2.3813 |

Two significant figures were lost in the subtraction of  $2bc \cos A$  from  $b^2 + c^2$ . Hence the lack of precision in the value of  $a$ .

Given  $b = .049$ ,  $c = .045$ ,  $A = 143^\circ$ ;  
find the remaining parts.

$$c_1 = b \cos A, \quad c_2 = c - c_1,$$

$$\tan B = \frac{c_1}{c_2} \tan A,$$

$$a = c_2 \sec B.$$

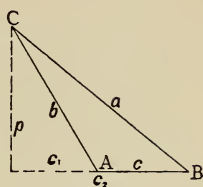


FIG. 47.

|                    |               |                 |
|--------------------|---------------|-----------------|
|                    | $L b$         | 8.6902          |
| $c = .045$         | $L \cos A$    | 9.9023 <i>n</i> |
| $c_1 = - .03913$   | $L c_1$       | 8.5925 <i>n</i> |
| $c_2 = .08413$     | $L c_2$       | 8.9250          |
|                    |               | 9.6675 <i>n</i> |
| $A = 143^\circ$    | $L \tan A$    | 9.8771 <i>n</i> |
| $B = 19^\circ 19'$ | $L \tan B$    | 9.5446          |
| $162^\circ 19'$    | $\log \sec B$ | 0.0252          |
| $C = 17^\circ 41'$ | $L c_2$       | 8.9250          |
| $a = .08917$       | $L a$         | 8.9502          |

Given  $b = .049$ ,  $c = .045$ ,  $A = 143^\circ$ ; find  $a$ .

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}.$$

|         |                |                 |
|---------|----------------|-----------------|
|         | $\log 2$       | 0.3010          |
|         | $L b$          | 8.6902          |
|         | $L c$          | 8.6532          |
|         | $L \cos A$     | 9.9023 <i>n</i> |
| .002401 | $L b^2$        | 7.3804          |
| .002025 | $L c^2$        | 7.3064          |
| .003521 | $L 2bc \cos A$ | 7.5467 <i>n</i> |
| .007947 | $L a^2$        | 7.9002          |
| .08915  | $L a$          | 8.9501          |

Given the three sides. —  $a = 704$ ,  $b = 302$ .  $c = 670$ .

Find the angles and the area.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}; \cot \frac{1}{2}A = \frac{s-a}{r}, \text{ etc.}; K = rs.$$

|             |     |                          |               |
|-------------|-----|--------------------------|---------------|
| 704         | 134 | $\log s - a$             | 2.1271        |
| 302         | 536 | $\log s - b$             | 2.7292        |
| 670         | 168 | $\log s - c$             | 2.2253        |
| <u>1676</u> |     |                          | <u>7.0816</u> |
| 868         | 868 | $\log s$                 | 2.9232        |
|             |     | $\log r^2$               | 4.1584        |
|             |     | $\log r$                 | 2.0792        |
| 83° 42'     |     | $\log \cot \frac{1}{2}A$ | 0.0479        |
| 25° 14'     |     | $\log \cot \frac{1}{2}B$ | 0.6500        |
| 71° 4'      |     | $\log \cot \frac{1}{2}C$ | 0.1461        |
| 100,550     |     | $\log K$                 | 5.0024        |

**Exercises.** — Assume three values in any row of the following table and compute the remaining values in that row.

| $a$  | $b$  | $c$  | $A$      | $B$     | $K$    |
|------|------|------|----------|---------|--------|
| 6.82 | 5.20 | 3.16 | 106° 47' | 46° 53' | 7.864  |
| .317 | .533 | .510 | 35° 18'  | 76° 19' | .07854 |
| 28.9 | 60.1 | 71.2 | 23° 32'  | 56° 9'  | 854.3  |
| 1.98 | 2.02 | 3.16 | 37° 22'  | 38° 16' | 1.937  |
| 312  | 109  | 229  | 131° 25' | 15° 11' | 9360   |
| 14.8 | 17.5 | 15.3 | 53° 8'   | 71° 5'  | 107.1  |
| 1.75 | 1.62 | 1.19 | 67° 23'  | 73° 44' | 99.96  |
| 145  | 119  | 156  | 61° 56'  | 46° 24' | 8190   |
| 11.6 | 40.4 | 48.0 | 11° 25'  | 43° 36' | 192.0  |

**41.** Whenever it is possible, by means of the ruler and compasses, to obtain the geometric construction of a problem, *the trigonometric solution is always obtainable by following the order of the construction.* It occasionally happens, however, that the geometric construction is not possible. When this occurs, the trigonometric solution produces an equation of a higher degree than the second. In many such cases, the method of solution by trial involves the least labor.

As an illustration, consider the following problem:

A ship is sailing in the direction N.  $35^\circ$  W. When at  $B$ , two lighthouses are sighted, one due north, the other due west. After sailing 3 miles, the ship was equally distant from the lighthouses; sailing 1 mile further on the same course, the ship and the lighthouses are in the same straight line. Find  $AC$ , the distance between the lights.

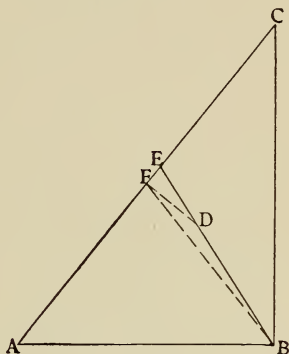


FIG. 48.

From  $D$ , the second position of the ship, drop a perpendicular to  $F$ , the middle of  $AC$ . Denote angle  $A$  by  $x$ ; then the geometry of the figure gives: angle  $BFE = 2x$ ,  $BEF = 125^\circ - x$ ,  $BFD = 2x - 90^\circ$ , and  $BDF = 215^\circ - x$ .

Apply (19) to triangles  $BFD$  and  $BFE$ :

$$BF = 3 \csc(2x - 90^\circ) \sin(215^\circ - x) = 4 \csc 2x \sin(125^\circ - x),$$

$$\text{or} \quad 3 \sec 2x \cos(125^\circ - x) = 4 \csc 2x \sin(125^\circ - x).$$

Multiply by  $\sin 2x \sec(125^\circ - x)$  and transpose:

$$3 \tan 2x - 4 \tan(125^\circ - x) = 0.$$

Now solve this by trial. Substituting  $0^\circ$  for  $x$  makes the left-hand member of the equation positive,  $90^\circ -$ ,  $45^\circ +$ ,  $60^\circ -$ ,  $50^\circ +$ ,  $52^\circ -$ , and so on;  $x = 51^\circ 7'$ . Hence

$$AC = 2 BF = 8 \csc 102^\circ 14' \sin 73^\circ 53' = 7.864 \text{ miles.}$$

In order to measure the distance between two inaccessible points,  $C$  and  $D$ , a base line  $AB$  was laid off equal to 583 feet ( $a$ ). At  $A$  angle  $CAD = 34^\circ$  and angle  $DAB = 47^\circ$  were measured. Also at  $B$ , the angle  $ABC = 44^\circ$  and angle  $CBD = 32^\circ$ . Find  $CD$ .

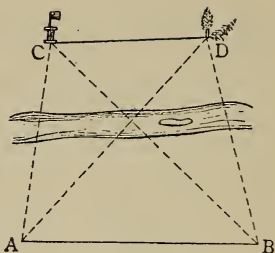


FIG. 49.

$$\left. \begin{aligned} AC &= a \csc ACB \sin ABC; \quad AD = a \csc ADB \sin ABD \\ CD^2 &= AC^2 + AD^2 - 2 AC \times AD \cos CAD \end{aligned} \right\} \quad (25)$$

|                  |                 |        |
|------------------|-----------------|--------|
|                  | $L \sin ABC$    | 9.8418 |
| $ACB = 55^\circ$ | $\log \csc ACB$ | 0.0866 |
|                  | $\log a$        | 2.7657 |
| $ADB = 57^\circ$ | $\log \csc ADB$ | 0.0764 |
| $ABD = 76^\circ$ | $L \sin ABD$    | 9.9869 |
|                  | $\log AC$       | 2.6941 |
| $AC^2 = 244500$  | $\log AD$       | 2.8290 |
| $AD^2 = 455000$  | $\log 2$        | 0.3010 |
| 699500           | $L \cos CAD$    | 9.9186 |
| 553000           |                 | 5.7427 |
|                  | $\log AC^2$     | 5.3882 |
|                  | $\log AD^2$     | 5.6580 |
| $CD^2 = 146500$  | $\log CD^2$     | 5.1658 |
| $CD = 382.7$     | $\log CD$       | 2.5829 |

In making an extended survey, it may happen that  $CD$ , Fig. 49, is a base of known length, the angles being measured from  $A$  and  $B$ . In this case, to find  $AB$ , assume the figure drawn to scale,  $AB$  being represented by 1 foot. In this reduced figure,  $CD$  may be computed; then the numerical value of  $AB$  equals the ratio of the known length of  $CD$  to the computed length of  $CD$  in the reduced figure.

The sides of a triangle are:  $a = 57$ ,  $b = 51$ ,  $c = 64$ . A point is taken in the side  $c$ , such that the segments of  $c$  are:  $h = 21$ ,  $k = 43$ . Find  $l$ , the distance of this point from  $C$ .

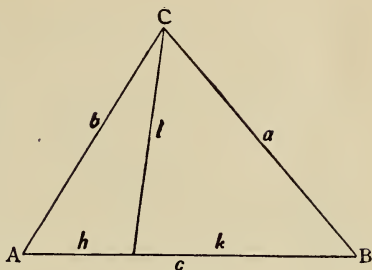


FIG. 50.

Denote the angle between  $h$  and  $l$  by  $D$ .

$$\text{By (20),} \quad l^2 + h^2 - 2hl \cos D = b^2;$$

$$\text{By (20),} \quad l^2 + k^2 - 2kl \cos (180^\circ - D) = a^2.$$

Multiply the first by  $k$ , the second by  $h$ , and add :

$$cl^2 + chh = a^2h + b^2k. \quad (26)$$

Substituting,  $l$  is found to be 43.71.

The top of a lighthouse, known to be 100 feet ( $h$ ) high, is just seen in the horizon. What is the distance ( $b$ ) of the lighthouse from the observer?

It is known that near the earth's surface light does not move in a horizontal line; on account of refraction, a ray, which seems horizontal, moves in a curve, which is nearly the arc of a circle whose radius is 7 times the earth's radius.

In (26) put  $R$  for  $h$ ,  $6R$  for  $k$ ,  $7R$  for  $a$  or  $c$ ,  $R + h$  for  $l$ . Then

$$6b^2 = 7(2Rh + h^2).$$

In the problem under consideration  $h^2$  is very small compared to  $2Rh$ , and may be neglected. Now  $h$  is usually given in feet, and  $b$  is desired in miles. Neglect  $h^2$  and multiply the left side of the equation by 5280 :

$$31680 b^2 (\text{miles}) = 14 R \times (h \text{ in feet}).$$

Put 3960 for  $R$  and solve for  $b$ .

$$b (\text{in miles}) = 1.323 \sqrt{h (\text{in feet})}. \quad (27)$$

Substituting,  $b = 13.23$  miles.

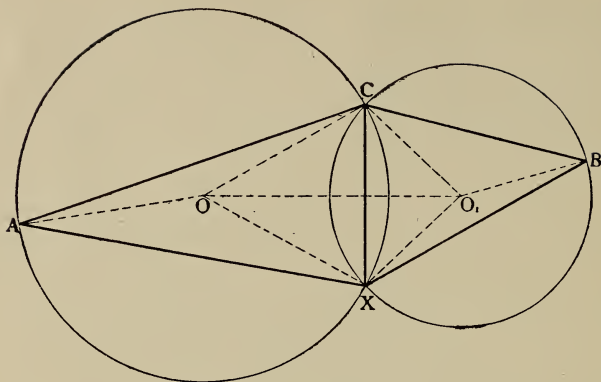


FIG. 51.

**Snell's Problem.** — On a coast are three prominent objects,  $A$ ,  $C$ , and  $B$ .  $CB = 1100$  feet ( $a$ ),  $AC = 1800$  feet ( $b$ ), and the angle  $ACB = 150^\circ$  ( $C$ ). A buoy is placed at a point  $X$ , and from a boat at the buoy the horizontal angles  $AXC = 80^\circ$  ( $D$ ) and  $CXB = 60^\circ$  ( $E$ ) are measured. Find the distance from  $X$  to  $C$ .\*

The following values are obtained by following the geometric construction: angle  $AOC = 2D$  (same arc),

angle  $BO_1C = 2E$ , angle  $OCO_1 = C + D + E - 180^\circ$ .

Hence  $OC = \frac{1}{2}a \csc D$ , and  $O_1C = \frac{1}{2}b \csc E$ .

Now denote angle  $COO_1 = XAC$  by  $A$ . From (24),

$$\left. \begin{aligned} \cot A &= -\frac{a}{b} \csc D \sin E \csc (C + D + E) - \cot (C + D + E) \\ CX &= a \csc D \sin A \end{aligned} \right\} \quad (28)$$

---

\* By many writers the credit has been given to Pothenot for the suggestion of this method of determining the position of a point by means of angles subtended by segments joining three known points. The problem, however, was proposed and solved in the year 1617 by Willebrord Snell, the discoverer of the law of the refraction of light. John Collins gave a solution of the problem in the Philosophical Transactions for 1671; Pothenot's solution was in 1692.

terms of the data,  $CX = \frac{\text{area } OCO_1}{OO_1}$

$$= \frac{ab \csc D \csc E}{2\sqrt{a^2 \csc^2 D + b^2 \csc^2 E + 2ab \csc D \csc E \cos(C + D + E)}}$$

The computation by (28) proceeds thus :

|                         |                 |                                       |                 |              |        |
|-------------------------|-----------------|---------------------------------------|-----------------|--------------|--------|
| $\log (-a)$             | 3.0414 <i>n</i> | .5719                                 | $\log a \csc D$ | 3.0480       |        |
| $\log \csc D$           | 0.0066          | $\text{nat cot } 290^{\circ} - .3640$ | $L \sin A$      | 9.8634       |        |
| $L \sin E$              | 9.9375          | $\text{nat cot } A$                   | .9359           | $\log CX$    | 2.9114 |
| $\log \csc 290^{\circ}$ | 0.0270 <i>n</i> | $A = 46^{\circ} 54'$                  |                 | $CX = 815.5$ |        |
|                         | <u>3.0125</u>   |                                       |                 |              |        |
| $\log b$                | <u>3.2553</u>   |                                       |                 |              |        |
|                         | <u>9.7572</u>   |                                       |                 |              |        |

In laying out a railroad along the seashore, the distances  $CB$  and  $DE$  were measured ( $C, B, D$ , and  $E$  all in a straight line); also the angles  $CAB, BAD$ , and  $DAE$  were measured. It was found that  $CB = 2200$  ( $a$ ),  $DE = 2300$  ( $b$ ),  $CAB = 21^\circ 30'$  ( $A_1$ ),  $BAD = 20^\circ 26'$  ( $A_2$ ), and  $DAE = 22^\circ 40'$  ( $A_3$ ); find  $BD$ .

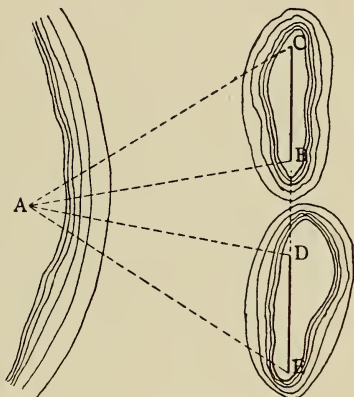


FIG. 52.

Denote  $BD$  by  $x$ . Then

In triangle  $ADC$ ,  $x + a = AC \csc D \sin(A_1 + A_2)$ .

In triangle  $AEB$ ,  $x + b = AE \csc B \sin(A_2 + A_3)$ .

In triangle  $ABC$ ,  $AC = a \csc A_1 \sin B$ .

In triangle  $AED$ ,  $AE = b \csc A_3 \sin D$ .

The product of these four equations is

$$(x+a)(x+b) = ab \csc A_1 \csc A_3 \sin(A_1 + A_2) \sin(A_2 + A_3). \quad (29)$$

This quadratic equation gives  $x = 1800$ .

There are two observatories on the same meridian:  $A$  in Sweden, latitude  $65^{\circ} 30' \text{ N.}$  ( $l$ );  $B$  at the Cape of Good Hope,

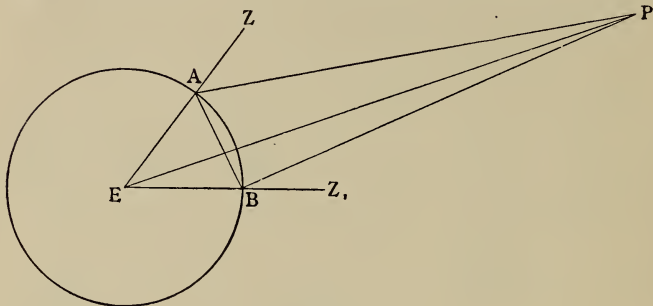


FIG. 53.

latitude  $34^{\circ} 20' \text{ S.}$  ( $-l_1$ ). When the moon passed this meridian, the angles  $ZAP = 62^{\circ} 20'$  ( $z$ ) and  $Z_1BP = 39^{\circ} 20'$  ( $z_1$ ) were measured. The radius of the earth being 3960 miles, find the distance from the moon to the earth's center.

The angles of the quadrilateral  $AEBP$  are readily found: at  $E$ , the angle is  $l - l_1$ ; at  $A$ , it is  $180^{\circ} - z$ ; at  $P$ , it is  $z + z_1 - l + l_1$ . Also the angle  $ABP$  is  $90^{\circ} + \frac{1}{2}(l - l_1 - 2z_1)$ .

In triangle  $AEB$ ,  $AB = 2 AE \sin \frac{1}{2} AEB$ .

In triangle  $ABP$ ,  $AP = AB \csc APB \sin ABP$ .

Denote  $AE$  by  $r$ ,  $AP$  by  $v$ ,  $EP$  by  $x$ , angle  $AEP$  by  $w$ .

$$\left. \begin{aligned} v &= 2r \sin \frac{1}{2}(l - l_1) \csc(z + z_1 - l + l_1) \cos \frac{1}{2}(l - l_1 - 2z_1) \\ \cot w &= \frac{r}{v} \csc z + \cot z \\ x &= v \csc w \sin z \end{aligned} \right\} (30)$$

|                           |        |                     |        |                           |        |
|---------------------------|--------|---------------------|--------|---------------------------|--------|
| $\log 2$                  | 0.3010 | $L \frac{r}{v}$     | 8.2257 | $\log v$                  | 5.3720 |
| $\log r$                  | 3.5977 | $\log \csc z$       | 0.0122 | $\log \csc w$             | 0.0139 |
| $L \sin 49^{\circ} 55'$   | 9.8837 |                     | 8.2379 | $L \sin z$                | 9.9878 |
| $\log \csc 1^{\circ} 30'$ | 1.6332 |                     | .0173  |                           | 5.3737 |
| $L \cos 25^{\circ} 15'$   | 9.9564 | $\text{nat cot } z$ | .2401  | $x = 236,400 \text{ mi.}$ |        |
| $\log v$                  | 5.3720 | $\text{nat cot } w$ | .2574  |                           |        |

## PROBLEMS.

**42. 64.** Solve Problem 10 by logarithms.

**65.** Solve Problem 11 by logarithms.

**66.** Solve Problem 12 by logarithms.

**67.** Solve Problem 13 by logarithms.

**68.** Solve Problem 14 by logarithms.

**69.** Solve Problem 15 by logarithms.

**70.** To measure the distance between two points  $A$  and  $C$  ( $C$  inaccessible), a base line  $AB = 693$  feet was measured from  $A$ . At  $A$ , the angle subtended by  $BC$  was  $68^\circ 29'$ ; and at  $B$ , the angle subtended by  $AC$  was  $66^\circ 7'$ . Find the distance  $AC$ .

**71.** To find the distance  $AB$  across a pond, a third point  $C$  was taken.  $AC = 735$  yards,  $BC = 840$  yards, and the angle  $ACB = 55^\circ 40'$  were measured. Find  $AB$ .

**72.** From a window on a level with the bottom of a steeple, the angle of elevation of the top is  $40^\circ$ . From another window 18 feet directly above the former the angle of elevation is  $37^\circ 30'$ . Required the height of the steeple.

**73.** A tower 50 feet high is situated on a cliff over the sea. From the bottom of the tower, the angle of depression of a ship at anchor is found to be  $25^\circ 17'$ ; and from the top the angle is  $31^\circ 12'$ . Find the height of the cliff and the direct distance of the ship from the top of the tower.

**74.** A frigate 10 miles S.W. of a harbor sees a ship sail from it in the direction S.  $70^\circ$  E. at the rate of 9 miles an hour. In what direction and at what rate must the frigate sail in order to come up with the ship in two hours?

75. A person is at the bank of a river directly opposite a tower on the other side. Going backward from the bank, up a slope making an angle of  $42^\circ$  to the horizontal, he measures an oblique distance of 528 feet. He there finds the angle of depression of the top of the tower is  $90^\circ$ , and of the base is  $27^\circ$ . Required the height of the tower and the breadth of the river.

76. To measure the breadth of a stream, a base  $AB$  was chosen = 490 yards long, parallel to the bank and 50 yards from it. A tree  $C$  stands on the other bank of the river. The angle  $CAB$  was found to be  $62^\circ 37'$  and  $CBA$   $40^\circ 28'$ . What was the breadth of the stream?

77. At the foot of a mountain the elevation of the summit =  $32^\circ 57'$ . A person measures 775.3 feet along the slope toward the summit, making an angle of  $7^\circ 15'$  with the horizontal plane; at this point, the elevation of the summit was  $60^\circ 32'$ . Find the height of the mountain.

78. A tower stands on a hill whose slope is uniformly  $26^\circ 37'$ . An observer at the foot of the hill finds that the tower subtends an angle of  $10^\circ 22'$ . Measuring *horizontally* up the slope a distance of 200 feet, he observes the tower subtends an angle of  $20^\circ 30'$ . Find the height of the tower.

79. A tower is situated on a hill. At a point in the horizontal plane through the foot of the hill, the angle of elevation of the top of the hill is  $40^\circ$  and of the top of the tower is  $51^\circ$ . Measuring in a line directly away from the hill a distance of 180 feet, the angle to the top of the tower is found to be  $33^\circ 45'$ . Find the height of the tower.

80. From a hill 200 feet above the sea, the top of a mast, known to be 150 feet above water, was just seen in the horizon. How far distant is it?

81. A body is thrown with a velocity of 25 feet per second horizontally from the window of a railway carriage moving at the rate of 30 miles an hour; the direction in

which the body is thrown makes an angle of  $30^\circ$  with the rear of the train. Find the direction of the vertical plane in which the body moves.

**82.** A man traveling at the rate of 6 miles an hour on a road that went due east observed that the wind struck him from the northeast. But having occasion to stop, he found that it actually came from the direction N.  $35^\circ$  E. Find the velocity of the wind.

**83.** A ship's apparent course is N.  $33^\circ 45'$  E., 8 knots an hour; but the tide sets her S.  $67^\circ 30'$  E. at the rate of 3 knots an hour. What is her true course and rate of progress?

**84.** Two forces,  $P = 6.2$  lbs. and  $Q = 10.9$  lbs., making an angle of  $112^\circ 4'$ , act at the same point on a body. Find the magnitude of the resultant and the angle which it makes with the force  $P$ .

**85.** Three forces in a plane acting on a body produce equilibrium. The greatest is 5 lbs., the smallest 3 lbs. The angle between the 3 lb. force and the unknown force is  $79^\circ 43'$ . Find the third force.

**86.** On the same meridian are two places whose difference of latitude is  $68^\circ 13'$ . From these places, the zenith distances of the moon at its culmination were found to be  $25^\circ$  and  $44^\circ 18'$ , respectively. The radius of the earth being 3960 miles, find the distance of the moon from the earth's center.

**87.** In measuring a distance  $AD$ , a portion of it ( $BC$ ) between  $A$  and  $D$  is inaccessible.  $AB = 244$ ,  $CD = 520$ . From a point  $E$  outside of  $AD$ , angles were measured:  $AEB = 82^\circ 8'$ ,  $BEC = 21^\circ 2'$ ,  $CED = 40^\circ 14'$ . Find  $BC$ .

**88.** A man walking along a straight road observes the angle of elevation of a tower to be  $28^\circ 40'$ . At a point 57 feet further on, the angle of elevation of the same tower is  $48^\circ 42'$ . And 143 feet beyond that, the angle is  $33^\circ 10'$ . Find the height of the tower.

89. From a ship, a rock is seen in the direction N.  $11^{\circ} 48'$  E. The ship sails due southeast 5 miles when the direction of the rock is N.  $1^{\circ} 36'$  E. Find the distance from the rock to each position of the ship.

90. The sides of a triangle are three consecutive whole numbers; and the greatest angle is double the least ( $A$ ). Find the average side ( $c$ ).

91. The distance between the centers of two circles is 238 ( $c$ ). The outer common tangents make with each other an angle of  $36^{\circ} 8'$  ( $A$ ). The inner common tangents make with each other an angle of  $104^{\circ} 12'$  ( $B$ ). Find the length of the radius of the larger circle.

92. The distances between three points  $A, B, C$  are as follows:  $AC = 1200$  feet,  $BC = 1320$  feet, and  $AB = 1400$  feet. An observer at  $X$  in the plane  $ABC$  finds angle  $AXC = 25^{\circ} 32'$  and  $BXC = 40^{\circ}$ . Find the distances of  $X$  from the points  $A, B$ , and  $C$ , if  $C$  and  $X$  are on different sides of  $AB$  and angle  $AXB = AXC + CXB$ .

93. From the top of a cliff 300 feet high, the altitude of the sun was observed to be  $70^{\circ}$ , the angle of elevation of a balloon was  $52^{\circ} 12'$ , the angle of depression of its shadow on the sea was  $68^{\circ} 14'$ . The sun is in front of the observer, and the cliff, sun, balloon, and shadow are all in the same vertical plane. Find the height of the balloon above the sea.

94. From a bluff 100 feet above the surface of a lake, the angle of elevation of a balloon is  $48^{\circ} 20'$ , and the angle of depression of its image reflected from the surface of the water is  $39^{\circ} 50'$ . Find the height of the balloon above the lake.

95. From the deck of a ship sailing due east, two conspicuous headlands are observed to bear  $63^{\circ}$  and  $24^{\circ}$ , respectively, to the northward of the ship's course. After

sailing 8 miles, the corresponding angles were found to be  $150^\circ$  and  $98^\circ$ . Find the distance of the headlands from one another.

96. Calculate the distance of two inaccessible points  $A$  and  $B$ , knowing a base  $CD = 600$  feet, the angle  $BCD = 40^\circ$ ,  $ACD = 69^\circ$ ,  $ADC = 38^\circ 30'$ , and the angle  $BDC = 70^\circ 30'$ .

97. Three points  $A$ ,  $C$ , and  $B$  being given on a chart of a coast, it is desired to determine the position of a fourth point  $D$ .  $AC = 1000$  feet,  $BC = 850$  feet, angle  $ACB = 114^\circ 40'$ , angle  $ADC = 46^\circ 17'$ , angle  $CDB = 30^\circ 9'$ . Find  $CD$ . [ $CD$  is between  $AD$  and  $BD$ ;  $C$  and  $D$  are on opposite sides of  $AB$ .]

98. From the top of a lighthouse 103.5 feet above the sea, the horizontal angle between two boats was found to be  $41^\circ 20'$ , and their angles of depression were  $13^\circ 23'$  and  $17^\circ 41'$  respectively. Find the distance between the boats.

99. A balloon is observed from two stations 1 mile apart. At the first station, the horizontal angle of the balloon and the other station  $= 82^\circ$ , and the elevation of the balloon  $= 12^\circ$ . At the second station, the horizontal angle between the first station and the balloon  $= 65^\circ$ . Find the height of the balloon.

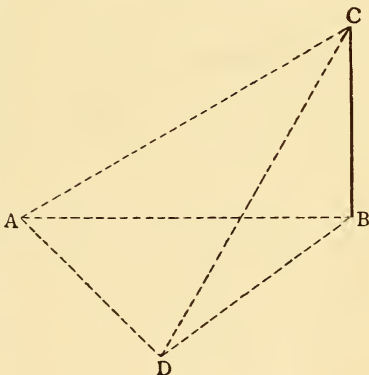


FIG. 54.

100. To determine the height  $BC$  of a tower, two points  $A$  and  $D$  90 feet apart were taken in the horizontal plane through its base. The angle  $BAD = 56^\circ 33'$ ,  $BAC = 65^\circ 23'$ ,  $BDC = 56^\circ 26'$ . Find  $BC$ .

**101.** To measure the breadth of a river  $AD$ , a point  $B$  was taken in  $AD$  prolonged. From  $B$  a base line  $BC = 400$  feet was laid off, making an angle of  $95^\circ 16'$  with  $AB$ . At  $C$ , the angles  $BCA = 52^\circ 48'$  and  $BCD = 24^\circ 39'$  were measured. Find  $AD$ .

**102.** From three points  $A, D, B$  in a horizontal straight line,  $AD = 500$  feet,  $DB = 300$  feet, the angles of elevation of a tower were measured, and found to be  $9^\circ 30'$ ,  $51^\circ 14'$ , and  $27^\circ 51'$ , respectively. Find the height of the tower.

**103.** Three points  $A, D, B$  lie in a straight line.  $AD = 232$ ,  $DB = 80$ . To locate a fourth point  $C$  in the horizontal plane of these three, the angles  $ACD = 85^\circ 12'$  and  $DCB = 46^\circ 13'$  were measured. Find  $CA, CD, CB$ .

**104.** The pedestal of a monument is 9 feet high, the column is 8 feet high, and the statue is 10 feet high. A person is in such a position that he sees the three parts subtending the same angle. What angle with the monument is made by the line joining his eye and the base of the monument; and what is the height of his eye above the base?

**105.** Within an angle  $A = 73^\circ 44'$ , a point is chosen whose distances from the sides of the angle are 6 inches and 8 inches. A line is drawn through this point so that the portion included between the sides of the angle is bisected at the point. Find the area of the triangle thus formed.

**106.** The two non-parallel sides of a trapezoid are 7.5 feet and 6.3 feet. The angle formed by their continuations  $= 41^\circ 21'$ . The base of the trapezoid is 10 feet. Find its area.

**107.** From two vertices of an equilateral triangle, whose side is 234 feet, two bodies begin to move simultaneously toward the third vertex; one with a velocity of 4 feet per second, the other with a velocity of 3 feet per second. When will the distance between the moving points be equal to the altitude of the triangle?

**108.** Calculate the radius of an inaccessible cylindrical tower from the following data: A base line  $AB$  is measured = 70 feet. The angles formed with the base line by the pair of tangents are: From  $A$ ,  $60^\circ$  and  $20^\circ$ ; and from  $B$ ,  $75^\circ$  and  $25^\circ$ .

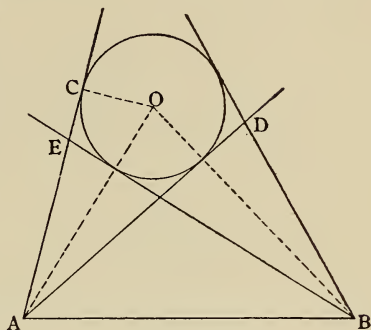


FIG. 55.

**109.** At some distance from a circular pond, a base line  $AB = 250$  feet is measured. At  $A$  the angles between the base line and the tangents to the pond are  $71^\circ 12'$  and  $54^\circ 12'$ . At  $B$ , the angle between the base line and the tangent to the right-hand side of the pond is  $57^\circ$ . Find the radius of the pond.

**110.** At some distance from a circular pond, a base line  $AB = 75$  yards is measured. At  $A$  the angles made by the base line and the tangents to the pond are  $72^\circ 58'$  and  $44^\circ 2'$ . At  $B$  the angle between the base line and the tangent to the left side of the pond is  $38^\circ 20'$ . Find the area of the pond.

**111.** To find the height of a hill, three points,  $A$ ,  $B$ , and  $C$ , not in a straight line, were found, such that at each of the three points the angle of elevation of the summit of the hill was the same; viz.  $52^\circ 40'$ .  $CB$  was measured, and found to be 121.6 feet, and angle  $BAC = 15^\circ 25'$ . Find the height of the hill.

## CHAPTER VI.

### PROPERTIES OF TRIANGLES.

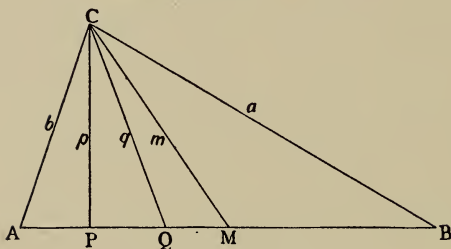


FIG. 56.

**43. The altitudes.** —  $p = a \sin B = b \sin A$ . Similar expressions hold for the other altitudes.

If the three altitudes are known, the triangle can be solved. Denote the altitudes by  $p_1$ ,  $p_2$ , and  $p_3$ . Then

$$a = \frac{2K}{p_1}, \quad b = \frac{2K}{p_2}, \quad c = \frac{2K}{p_3}.$$

Substitute in (21);

$$\cos A = \frac{\frac{1}{p_2^2} + \frac{1}{p_3^2} - \frac{1}{p_1^2}}{2} = \frac{p_3^2 p_1^2 + p_1^2 p_2^2 - p_2^2 p_3^2}{2 p_1^2 p_2 p_3},$$

with similar expressions for  $\cos B$  and  $\cos C$ . Then

$$b = p_3 \csc A; \quad c = p_2 \csc A, \text{ etc.}$$

**The bisectors of the angles.** — Let  $CQ (= q)$  be the bisector of the angle  $ACB$ . Then

$$\text{area } BCQ + \text{area } QCA = \text{area } BCA;$$

$$\frac{1}{2} a q \sin \frac{1}{2} C + \frac{1}{2} b q \sin \frac{1}{2} C = \frac{1}{2} ab \sin C = ab \sin \frac{1}{2} C \cos \frac{1}{2} C.$$

$$\text{Whence} \quad q = \frac{2ab \cos \frac{1}{2}C}{a+b}.$$

Now angle  $PCQ = \frac{1}{2}(A - B)$ ; hence  $q = p \sec \frac{1}{2}(A - B)$ .

$$\text{Hence} \quad q = a \sin B \sec \frac{1}{2}(A - B).$$

Similar values can be obtained for the other bisectors.

$$AQ = b \csc AQC \sin \frac{1}{2}C; \quad QB = a \csc CQB \sin \frac{1}{2}C.$$

$$\text{Since } \csc AQC = \csc CQB, \quad \frac{AQ}{QB} = \frac{b}{a}.$$

$$\text{But } AQ + QB = c; \text{ hence } AQ = \frac{bc}{a+b} \text{ and } QB = \frac{ac}{a+b}.$$

**The medians.**—Let  $M$  be the middle point of  $AB$ . By (20),

$$a^2 = \frac{1}{4}c^2 + m^2 - cm \cos BMC;$$

$$b^2 = \frac{1}{4}c^2 + m^2 - cm \cos AMC.$$

Now  $\cos BMC + \cos AMC = 0$ . Add and solve for  $m$ .

$$m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

Similar relations hold for the other medians.

If the angles are known and the medians are desired, find the angle at  $M$  first. Denote the middle point of  $BC$  by  $M_1$ , the middle point of  $CA$  by  $M_2$ . Then by (24),

$$\begin{aligned} \cot AMC &= \frac{c \csc A}{2b} - \cot A \\ &= \frac{1}{2}(\cot B - \cot A). \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \cot BM_1A &= \frac{1}{2}(\cot C - \cot B), \\ \cot CM_2B &= \frac{1}{2}(\cot A - \cot C). \end{aligned}$$

$$\text{Check: } \cot AMC + \cot BM_1A + \cot CM_2B = 0.$$

If the three medians are known,  $m_1, m_2, m_3$ , the sides can be found. Their values can be obtained from these equations:

$$9a^2 = 8m_2^2 + 8m_3^2 - 4m_1^2,$$

$$9b^2 = 8m_3^2 + 8m_1^2 - 4m_2^2,$$

$$9c^2 = 8m_1^2 + 8m_2^2 - 4m_3^2.$$

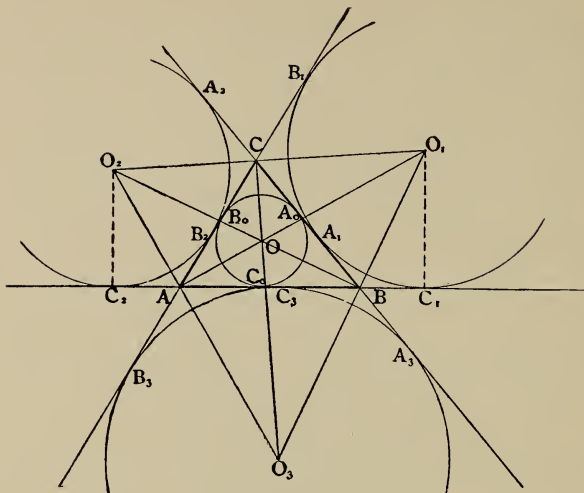


FIG. 57.

**44. The incircle and the excircles.** — Let  $O$  be the center of the incircle touching the sides of the triangle  $ABC$  in  $A_0$ ,  $B_0$ , and  $C_0$ ;  $O_1$ , the center of the excircle in angle  $A$  and  $A_1$ ,  $B_1$ ,  $C_1$ , its points of tangency, etc. Then

$$\begin{aligned} AC_1 &= AB_1 = BA_2 = BC_2 = CB_3 = CA_3 = s, \\ AC_0 &= AB_0 = BA_3 = BC_3 = CB_2 = CA_2 = s - a, \\ AC_3 &= AB_3 = BA_0 = BC_0 = CB_1 = CA_1 = s - b, \\ AC_2 &= AB_2 = BA_1 = BC_1 = CB_0 = CA_0 = s - c. \end{aligned}$$

Denote the radius of the incircle by  $r$ , the radii of the excircles by  $r_1$ ,  $r_2$ , and  $r_3$  respectively.

Values of the various segments in the figure will now be obtained in terms of the sides.

$$\begin{aligned} OC_0 &= OB \sin \frac{1}{2} B; \quad OB = a \sin \frac{1}{2} C \csc BOC (= 90^\circ + \frac{1}{2} A). \\ r &= a \sec \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C \\ &= b \sin \frac{1}{2} A \sec \frac{1}{2} B \sin \frac{1}{2} C \\ &= c \sin \frac{1}{2} A \sin \frac{1}{2} B \sec \frac{1}{2} C. \end{aligned}$$

Since  $O_1C_1 = O_1B \cos \frac{1}{2} B$ ,

and  $O_1B = a \csc CO_1B (= 90^\circ - \frac{1}{2} A) \sin O_1CB (= 90^\circ - \frac{1}{2} C)$ .

Also,  $O_1B = c \csc AO_1B (= \frac{1}{2} C) \sin \frac{1}{2} A$ .

$$\begin{aligned} r_1 &= a \sec \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C \\ &= b \sin \frac{1}{2} A \csc \frac{1}{2} B \cos \frac{1}{2} C \\ &= c \sin \frac{1}{2} A \cos \frac{1}{2} B \csc \frac{1}{2} C. \end{aligned}$$

Find  $O_2C_2 (= r_2)$  and  $O_3C_3 (= r_3)$  in a similar manner.

Since  $AC_1 = r_1 \cot \frac{1}{2} A$ .

$$\begin{aligned} s &= a \csc \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C \\ &= b \cos \frac{1}{2} A \csc \frac{1}{2} B \cos \frac{1}{2} C \\ &= c \cos \frac{1}{2} A \cos \frac{1}{2} B \csc \frac{1}{2} C. \end{aligned}$$

$$AC_0 = r \cot \frac{1}{2} A; \quad BC_0 = r \cot \frac{1}{2} B; \quad CA_0 = r \cot \frac{1}{2} C.$$

$$\begin{aligned} s - a &= a \csc \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C \\ &= b \cos \frac{1}{2} A \sec \frac{1}{2} B \sin \frac{1}{2} C \\ &= c \cos \frac{1}{2} A \sin \frac{1}{2} B \sec \frac{1}{2} C. \end{aligned}$$

$$\begin{aligned} s - b &= a \sec \frac{1}{2} A \cos \frac{1}{2} B \sin \frac{1}{2} C \\ &= b \sin \frac{1}{2} A \csc \frac{1}{2} B \sin \frac{1}{2} C \\ &= c \sin \frac{1}{2} A \cos \frac{1}{2} B \sec \frac{1}{2} C. \end{aligned}$$

$$\begin{aligned} s - c &= a \sec \frac{1}{2} A \sin \frac{1}{2} B \cos \frac{1}{2} C \\ &= b \sin \frac{1}{2} A \sec \frac{1}{2} B \cos \frac{1}{2} C \\ &= c \sin \frac{1}{2} A \sin \frac{1}{2} B \csc \frac{1}{2} C. \end{aligned}$$

$$AO = (s - a) \sec \frac{1}{2} A; \quad BO = (s - b) \sec \frac{1}{2} B; \quad CO = (s - c) \sec \frac{1}{2} C.$$

$$AO_1 = s \sec \frac{1}{2} A; \quad BO_1 = (s - c) \csc \frac{1}{2} B; \quad CO_1 = (s - b) \csc \frac{1}{2} C.$$

$$AO_2 = (s - c) \csc \frac{1}{2} A; \quad BO_2 = s \sec \frac{1}{2} B; \quad CO_2 = (s - a) \csc \frac{1}{2} C.$$

$$AO_3 = (s - b) \csc \frac{1}{2} A; \quad BO_3 = (s - a) \csc \frac{1}{2} B; \quad CO_3 = s \sec \frac{1}{2} C.$$

$$OO_1 = AO_1 - AO = a \sec \frac{1}{2} A; \quad OO_2 = b \sec \frac{1}{2} B; \quad OO_3 = c \sec \frac{1}{2} C.$$

$$O_2O_3 = AO_2 + AO_3 = a \csc \frac{1}{2} A; \quad O_3O_1 = b \csc \frac{1}{2} B; \quad O_1O_2 = c \csc \frac{1}{2} C.$$

Angle

$$O_3O_1O_2 = 90^\circ - \frac{1}{2} A; \quad O_1O_2O_3 = 90^\circ - \frac{1}{2} B; \quad O_2O_3O_1 = 90^\circ - \frac{1}{2} C.$$

Functions of the half-angles of a triangle.

$$\sin^2 \frac{1}{2} A = \frac{(s-b)(s-c)}{bc}; \quad \csc^2 \frac{1}{2} A = \frac{bc}{(s-b)(s-c)};$$

$$\tan^2 \frac{1}{2} A = \frac{(s-b)(s-c)}{s(s-a)}; \quad \cot^2 \frac{1}{2} A = \frac{s(s-a)}{(s-b)(s-c)};$$

$$\sec^2 \frac{1}{2} A = \frac{bc}{s(s-a)}; \quad \cos^2 \frac{1}{2} A = \frac{s(s-a)}{bc};$$

With similar relations for  $\frac{1}{2} B$  and  $\frac{1}{2} C$ .

**45. The circumcircle.** — Let  $F$  be the center of the circumcircle, and  $M$  the middle point of  $AB$ . Then since angle  $AFB$  at the center of the circle is twice the angle  $C$  on the circumference, angle  $AFM = C$ .

Now  $AF = AM \csc AFM$ . Therefore, changing letters,

$$R = \frac{1}{2} a \csc A = \frac{1}{2} b \csc B \\ = \frac{1}{2} c \csc C.$$

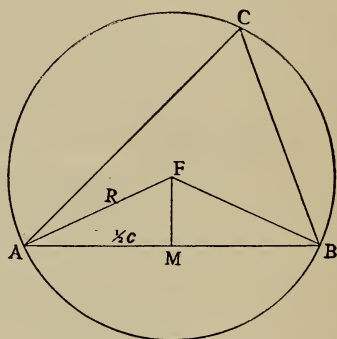


FIG. 58.

$$\text{Now } K = \frac{1}{2} bc \sin A = rs. \quad \text{Hence } R = \frac{abc}{4K} = \frac{abc}{4rs}.$$

Also,  $R \sin A = \frac{1}{2} a$ ,  $R \sin B = \frac{1}{2} b$ ,  $R \sin C = \frac{1}{2} c$ ; the sum and product of which give

$$R(\sin A + \sin B + \sin C) = s, \\ R^3 \sin A \sin B \sin C = \frac{1}{8} abc = \frac{1}{2} RK.$$

**46. Relations between the sum or difference of two sides and the sum or difference of two angles.**

Let  $ABC$  be the triangle under consideration. With  $C$  as a center and radius equal to  $CA$  ( $b$ ), describe a semicircle, cutting  $BC$  in  $D$  and  $E$ . Then  $BD = a - b$  and  $BE = a + b$ .

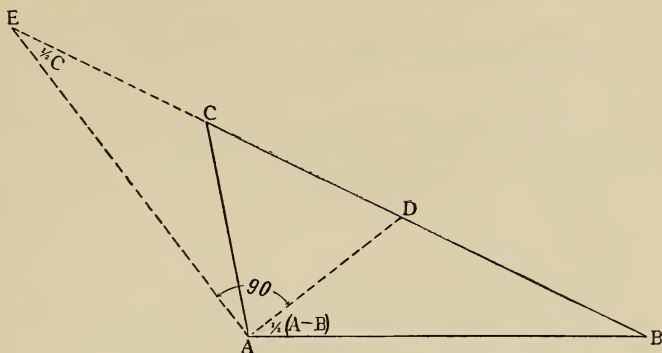


FIG. 59.

Angle  $DAE = 90^\circ$  (in semicircle), angle  $E$  (on circumference)  $= \frac{1}{2} C$ , angle  $BDA$  (exterior)  $= 90^\circ + \frac{1}{2} C$ , angle  $DAB = 180^\circ - B - (90^\circ + \frac{1}{2} C) = \frac{1}{2} (A - B)$ .

In triangle  $ABD$ , by (19),  $AB \sin DAB = DB \sin ADB$ .

$$c \sin \frac{1}{2} (A - B) = (a - b) \cos \frac{1}{2} C. \quad (31)$$

In triangle  $AEB$ , by (19),  $AB \sin EAB = EB \sin E$ .

$$c \cos \frac{1}{2} (A - B) = (a + b) \sin \frac{1}{2} C. \quad (32)$$

The quotient of (31) and (32) reduces to

$$(a + b) \tan \frac{1}{2} (A - B) = (a - b) \tan \frac{1}{2} (A + B).$$

The product and difference of the squares reduce to

$$c^2 \sin (A - B) = (a^2 - b^2) \sin C.$$

$$c^2 \cos (A - B) = 2ab - (a^2 + b^2) \cos C.$$

Eliminate  $a^2$  by (20), and  $a$  and  $b$  by (19).

$$\sin (A - B) = \sin C - 2 \sin B \cos A, \quad (33)$$

$$\cos (A - B) = 2 \sin A \sin B - \cos C. \quad (34)$$

The other functions of  $A - B$  can then be found. Since  $A + B = 180^\circ - C$ , any function of  $A + B$  can be expressed in terms of functions of  $C$ .

Also, since  $B + \frac{1}{2} C = 90^\circ - \frac{1}{2} (A - B)$ , any function of  $B + \frac{1}{2} C$  is equal to the cofunction of  $\frac{1}{2} (A - B)$ .

**47. General method of solving a triangle, given three segments in or about the triangle.**—Suppose that to solve the triangle there is given any three segments ( $R, r, r_1, s, s-a$ , etc.). Each of these segments may be expressed in terms of the sides and functions of the half angles; these functions may be expressed in terms of the sides. Hence the sides may be found by solving the set of equations thus obtained.

**Relations between the segments.**—By using values of any two of these segments a relation between the segments may be obtained. For example, find  $r$  and  $s$  in terms of  $R$ .

$$r = a \sec \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C.$$

$$a = 2 R \sin A = 4 R \sin \frac{1}{2} A \cos \frac{1}{2} A.$$

Hence  $r = 4 R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C.$

Similarly,  $s = 4 R \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$

**48. Points of intersection of the circumcircle.**—Let triangle  $ABC$ , Fig. 60, be the triangle under consideration, the bisectors of the interior and exterior angles being drawn as in Fig. 57.

Also, let  $F$  be the center of the circumcircle of the triangle  $ABC$ . Since angle  $M'''CO$  is a right angle, it is inscribed in a semicircle; hence  $M'''F$  passes through  $K'''$ , the intersection of the circumcircle and  $CO_3$ . The lengths of  $O_2M'''$  and of  $O_3K'''$  will now be determined.

Since  $FCB = 90^\circ - A$  (see Fig. 58),

angle  $FCK''' = CK'''F = \frac{1}{2} C - (90^\circ - A) = \frac{1}{2} (A - B).$

$CM''' = 2 R \sin \frac{1}{2} (A - B), \quad CK''' = 2 R \cos \frac{1}{2} (A - B).$

But  $R = \frac{1}{4} c \sec \frac{1}{2} C \csc \frac{1}{2} C,$

$c \sin \frac{1}{2} (A - B) = (a - b) \cos \frac{1}{2} C,$

$c \cos \frac{1}{2} (A - B) = (a + b) \sin \frac{1}{2} C.$

$CM''' = \frac{1}{2} (a - b) \csc \frac{1}{2} C; \quad CK''' = \frac{1}{2} (a + b) \sec \frac{1}{2} C.$

$O_2M''' = \frac{1}{2} c \csc \frac{1}{2} C; \quad O_3K''' = \frac{1}{2} c \sec \frac{1}{2} C.$

Whence  $M'''$  and  $K'''$  are the middle points of  $O_1O_2$  and  $OO_3$ , respectively.

In the same manner it may be shown that  $M'$ ,  $K'$ ,  $M''$ ,  $K''$ , are the middle points of  $O_2O_3$ ,  $OO_1$ ,  $O_1O_3$ ,  $OO_2$ , respectively.

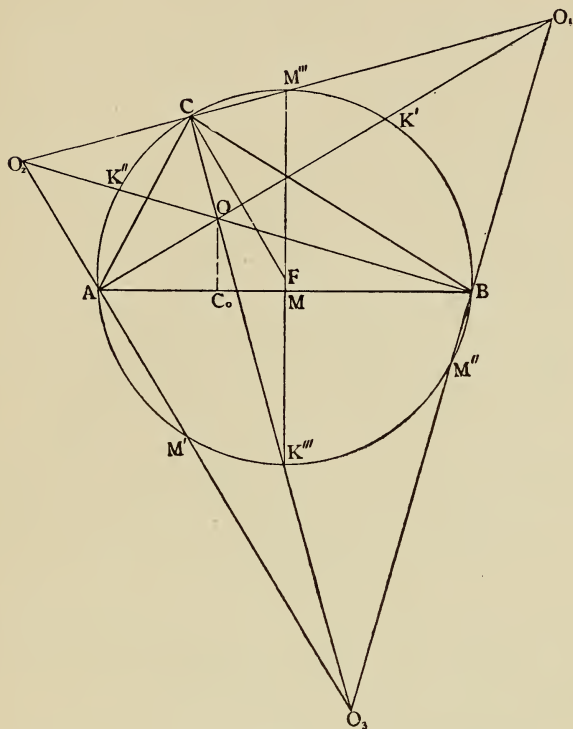


FIG. 60.

If  $OC_0(=r)$  is perpendicular to  $AB$ , the angle  $C_0OK'''$  is also equal to  $\frac{1}{2}(A - B)$ , and, therefore,  $K'''F$  is parallel to  $OC_0$ ; it is, therefore, perpendicular to  $AB$  through its middle point  $M$ . Hence,  $K'''$  is the middle point of the

arc  $AB$ ,  $K'$  is the middle point of the arc  $BC$ , and  $K''$  is the middle point of the arc  $CA$ .

The circumcircle of  $ABC$  is called the **nine-points circle** of the triangle  $O_1O_2O_3$ .

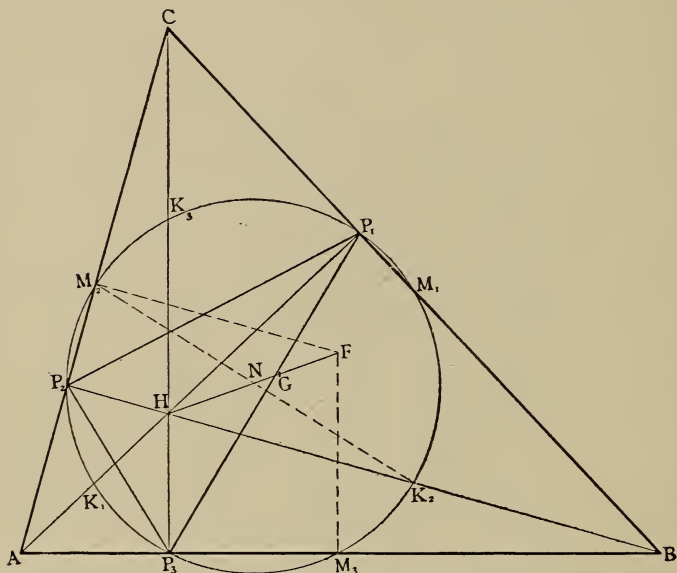


FIG. 61.

**49. The nine-points circle.** — Let  $ABC$  be the triangle under consideration. Draw the altitudes  $AP_1$ ,  $BP_2$ ,  $CP_3$ , meeting in  $H$ . Join  $P_1P_2$ ,  $P_2P_3$ ,  $P_3P_1$ .

In the triangle  $AP_3C$ ,  $AP_3 = b \cos A$ ,  
and in triangle  $AP_2B$ ,  $AP_2 = c \cos A$ .

Therefore the triangles  $AP_2P_3$  and  $ABC$  having a common angle and the including sides in each triangle proportional are similar; hence  $P_2P_3 = a \cos A$ .

In a similar manner  $P_3P_1 = b \cos B$  and  $P_1P_2 = c \cos C$ .

Also angle  $BP_1P_3 = CP_1P_2 = A$ , whence  $P_3P_1P_2 = 180^\circ - 2A$ ,  $P_1P_2P_3 = 180^\circ - 2B$ ,  $P_2P_3P_1 = 180^\circ - 2C$ .

The intersection  $H$  of the altitudes is called the **orthocenter** of the triangle  $ABC$ ; the triangle  $P_1P_2P_3$ , joining the feet of the altitudes, is called the **pedal triangle** of the triangle  $ABC$ . It should be noticed that the orthocenter of a triangle is the center of the incircle of the pedal triangle. **The circumcircle of the pedal triangle is the nine-points circle of the original triangle.**

All of the properties of any triangle  $ABC$  can be reduced to properties of its pedal triangle  $P_1P_2P_3$  by substituting in any formula referring to the triangle  $ABC$ ,  $180^\circ - 2A$  wherever  $A$  occurs,  $180^\circ - 2B$  wherever  $B$  occurs,  $180^\circ - 2C$  wherever  $C$  occurs. After completing these substitutions, put  $a \cos A$  wherever  $a$  occurs,  $b \cos B$  wherever  $b$  occurs, and  $c \cos C$  wherever  $c$  occurs. For instance, make these substitutions in the value of  $R$ . The radius of the circumcircle of  $P_1P_2P_3$  or the nine-points circle of  $ABC$

$$\begin{aligned} &= \frac{1}{4} a \cos A \sec \frac{1}{2}(180^\circ - 2A) \csc \frac{1}{2}(180^\circ - 2A) \\ &= \frac{1}{4} a \csc A = \frac{1}{2} R. \end{aligned}$$

That is, the radius of the nine-points circle of any triangle is one-half the radius of the circumcircle of the triangle.

All other properties of the nine-points circle of  $O_1O_2O_3$  (circumcircle of  $ABC$ ) can be reduced to properties of the nine-points circle of  $ABC$  (circumcircle of  $P_1P_2P_3$ ) by the same substitutions.

If  $F$  is the center of the circumcircle (see Fig. 58),

$$K = \text{area } BFC + \text{area } CFA + \text{area } AFB;$$

that is,  $rs = \frac{1}{2} aR \cos A + \frac{1}{2} bR \cos B + \frac{1}{2} cR \cos C$ .

$$\text{Hence} \quad \frac{2s}{a \cos A + b \cos B + c \cos C} = \frac{2\pi R}{2\pi r}.$$

That is, the perimeter of the triangle is to the perimeter of the pedal triangle as the circumference of the circumcircle is to the circumference of the incircle.

**50. Abscissæ and ordinates of various centers.**—Count abscissæ from  $A$  toward  $B$ .

For  $F$ , the center of the circumcircle.

The abscissa is  $\frac{1}{2}c$  and the ordinate is  $\frac{1}{2}c \cot C$ .

For  $O$ , the center of the incircle.

The abscissa is  $s - a$  and the ordinate is  $(s - a) \tan \frac{1}{2}A$ .

For  $O_1$ , the center of the excircle in  $A$ .

The abscissa is  $s$  and the ordinate is  $s \tan \frac{1}{2}A$ .

For  $O_2$ , the center of the excircle in  $B$ .

The abscissa is  $-(s - c)$  and the ordinate is  $(s - c) \cot \frac{1}{2}A$ .

For  $O_3$ , the center of the excircle in  $C$ .

The abscissa is  $s - b$  and the ordinate is  $-(s - b) \cot \frac{1}{2}A$ .

For  $H$ , the intersection of the altitudes (orthocenter).

The abscissa is  $b \cos A$  and the ordinate is  $b \cos A \cot B$ .

For  $G$ , the intersection of the medians.

The abscissa is  $\frac{1}{3}(b \cos A + c)$  and the ordinate is  $\frac{1}{3}b \sin A$ .

For  $N$ , the center of the nine-points circle.

The abscissa is  $\frac{1}{2}b \cos A + \frac{1}{4}c$  and the ordinate is  $\frac{1}{2}b \sin A - \frac{1}{4}c \cot C$ .

To obtain the abscissa and ordinate of  $N$ , note that the angle  $BP_3N$  is  $90^\circ - (A - B)$ . Then these values can be deduced from (33) and (34).

To obtain the distance between any two of these points, subtract their abscissa and square the difference; subtract their ordinates and square the difference. The square root of the sum of the squares gives the required distance.

These distances can also be obtained from the triangle. Thus to find the

distance  $FH$ , find  $AH$ ,  $AF$ , angle  $FAH$ , and apply (20):

$$\begin{aligned} AH &= b \cos A \sec (90^\circ - B) && \text{(See Fig. 61)} \\ &= b \csc B \cos A = a \csc A \cos A = a \cot A. \end{aligned}$$

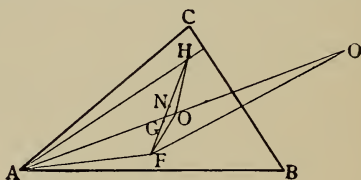


FIG. 62.

Since  $a = 2 R \sin A$ ,  $AH = 2 R \cos A$ .

Angle  $BAH = 90^\circ - B$ ;  $BAF = 90^\circ - C$ ;

hence  $\text{angle } FAH = C - B$ .

Hence  $FH^2 = R^2[1 + 4 \cos^2 A - 4 \cos(C - B) \cos A]$ .

In (34) put  $90^\circ - C$  for  $A$ ,  $90^\circ - B$  for  $B$ , and  $180^\circ - A$  for  $C$ .

$$\cos(C - B) = 2 \cos C \cos B + \cos A.$$

Hence  $FH^2 = R^2(1 - 8 \cos A \cos B \cos C)$ .

To find  $OF$  and  $O_1F$ .

$AO = r \csc \frac{1}{2} A$  and  $\text{angle } FAO = \frac{1}{2} A - (90^\circ - C) = \frac{1}{2} (C - B)$ .

$$OF^2 = R^2 + r^2 \csc^2 \frac{1}{2} A - 2 Rr \csc \frac{1}{2} A \cos \frac{1}{2} (C - B).$$

Since  $r = 4 R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$ ,

$$r^2 \csc^2 \frac{1}{2} A = 4 Rr \csc \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C = 2 Rr \frac{2(s-a)}{a}.$$

By (32),  $\csc \frac{1}{2} A \cos \frac{1}{2} (C - B) = \frac{b+c}{a}$ .

Substituting  $OF^2 = R^2 - 2 Rr$ .

Similarly,  $O_1F^2 = R^2 + 2 Rr_1$ .

**Exercises.** — Show that  $OH = 2 r^2 - 4 R^2 \cos A \cos B \cos C$ .

$$ON = \frac{1}{2} R - r.$$

$$O_1N = \frac{1}{2} R + r_1.$$

## PROBLEMS.

**51. 112.** The radius of the incircle of a triangle is 6;  $B = 38^\circ 53'$ ,  $C = 73^\circ 44'$ . A line is drawn across angle  $A$  parallel to the side  $BC$  and tangent to the incircle. In the small triangle thus formed, a circle is inscribed. Find the radius of this latter circle.

**113.** In the triangle of the last problem ( $r=6$ ,  $A=67^\circ 23'$ ), a small circle is drawn touching each side of angle  $A$ , and

also tangent externally to the incircle. Find the radius of this small circle.

114. In the same triangle ( $r = 6$ ,  $A = 67^\circ 23'$ ,  $B = 38^\circ 53'$ ), the incircle touches the sides at  $A_0$ ,  $B_0$ , and  $C_0$ . Find the radius of the incircle of the triangle  $A_0 B_0 C_0$ .

**Malfatti's problem.** — To inscribe in a triangle three circles, each touching the other two and also two sides of the triangle.\*

Let the sides of the triangle be  $a = 148$ ,  $b = 175$ ,  $c = 153$ , and the angles  $A = 53^\circ 8'$ ,  $B = 71^\circ 4'$ ,  $C = 55^\circ 48'$ .

Denote the radii of the Malfatti circles by  $x$ ,  $y$ , and  $z$ .

$$EF^2 = DE^2 - DF^2$$

$$= (x + y)^2 - (x - y)^2 = 4xy.$$

Hence

$$x \cot \frac{1}{2} A + 2\sqrt{xy} + y \cot \frac{1}{2} B = c.$$

Multiply by  $\sin \frac{1}{2} B \cos \frac{1}{2} B = \frac{1}{2} \sin B$ .

$$x \cot \frac{1}{2} A \sin \frac{1}{2} B \cos \frac{1}{2} B + 2 \sin \frac{1}{2} B \cos \frac{1}{2} B \sqrt{xy} + y \cos^2 \frac{1}{2} B = \frac{1}{2} c \sin B.$$

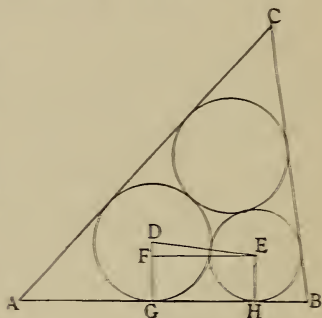
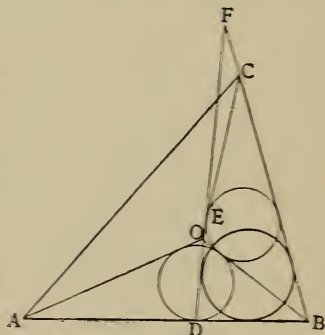


FIG. 63.

\* Steiner has given the following construction for this problem. Let  $O$  be the center of the incircle of  $ABC$ . Through the point of contact  $D$  of the incircle of  $AOB$ , draw a tangent to the incircle of  $BOC$ , meeting  $BC$  in  $F$ . The incircle of the triangle  $BDF$  is a Malfatti circle. Each of the other two Malfatti circles is constructed in a similar manner.



The coefficient of  $x$  equals  $\sin^2 \frac{1}{2} B + \frac{s-a}{a}$ ;

also  $c \sin B = p_1$ . Hence

$$(\sqrt{x} \sin \frac{1}{2} B + \sqrt{y} \cos \frac{1}{2} B)^2 = \frac{1}{2} p_1 - \frac{(s-a)x}{a}.$$

When  $y$  is changed to  $z$ ,  $B$  is changed to  $C$ .

$$(\sqrt{x} \sin \frac{1}{2} C + \sqrt{z} \cos \frac{1}{2} C)^2 = \frac{1}{2} p_1 - \frac{(s-a)x}{a}.$$

Equating and changing letters,

$$\sqrt{x} \sin \frac{1}{2} B + \sqrt{y} \cos \frac{1}{2} B = \sqrt{x} \sin \frac{1}{2} C + \sqrt{z} \cos \frac{1}{2} C.$$

$$\sqrt{y} \sin \frac{1}{2} C + \sqrt{z} \cos \frac{1}{2} C = \sqrt{y} \sin \frac{1}{2} A + \sqrt{x} \cos \frac{1}{2} A.$$

Adding these and arranging,

$$\frac{\sqrt{y}}{\sqrt{x}} = \frac{\cos \frac{1}{2} A - \sin \frac{1}{2} B + \sin \frac{1}{2} C}{\cos \frac{1}{2} B - \sin \frac{1}{2} A + \sin \frac{1}{2} C} = .936.$$

Substituting,  $x = 30$ ,  $y = 26.28$ ,  $z = 28.61$ .

**Properties of the Malfatti radii.**—In the equation\*

$$\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C,$$

put  $90^\circ - \frac{1}{2} A$  instead of  $A$ ,  $-\frac{1}{2} B$  instead of  $B$ , and  $180^\circ - \frac{1}{2} C$  instead of  $C$ ; then change  $A$  and  $B$ :

$$\cos \frac{1}{2} A - \sin \frac{1}{2} B + \sin \frac{1}{2} C = 4 \cos (45^\circ - \frac{1}{4} A) \cos \frac{1}{4} B \sin \frac{1}{4} C,$$

$$\cos \frac{1}{2} B - \sin \frac{1}{2} A + \sin \frac{1}{2} C = 4 \cos \frac{1}{4} A \cos (45^\circ - \frac{1}{4} B) \sin \frac{1}{4} C.$$

Hence 
$$\frac{\sqrt{y}}{\sqrt{x}} = \frac{\cos (45^\circ - \frac{1}{4} A) \cos \frac{1}{4} B}{\cos \frac{1}{4} A \cos (45^\circ - \frac{1}{4} B)}.$$

Now  $\cos (45^\circ - \frac{1}{4} A) = \cos 45^\circ (\cos \frac{1}{4} A + \sin \frac{1}{4} A), \dagger$

Also  $\cos (45^\circ - \frac{1}{4} B) = \cos 45^\circ (\cos \frac{1}{4} B + \sin \frac{1}{4} B).$

Hence

$$\sqrt{x} (1 + \tan \frac{1}{4} A) = \sqrt{y} (1 + \tan \frac{1}{4} B) = \sqrt{z} (1 + \tan \frac{1}{4} C).$$

\* Proved on next page.

† See equation (68), Appendix II, page 145.

**52. Relations between the angles of a triangle.** — When two relations connecting the same two segments are obtained, their comparison gives a relation between the angles of the triangle.

For example,  $R(\sin A + \sin B + \sin C) = s$ .

$$s = 4 R \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

$$\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

And since the only condition on  $A$ ,  $B$ , and  $C$  is that their sum is  $180^\circ$ , this equation is true for any three angles having this sum. Thus, apply it to the pedal triangle,

$$180^\circ - 2A \text{ for } A, \quad 180^\circ - 2B \text{ for } B, \quad 180^\circ - 2C \text{ for } C.$$

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

By adding and multiplying equations in (22),

$$\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \frac{3s - (a + b + c)}{r} = \frac{s}{r};$$

$$\cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C = \frac{(s-a)(s-b)(s-c)}{r^3} = \frac{r^2 s}{r^3} = \frac{s}{r}.$$

Hence

$$\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C.$$

In this, put

$$180^\circ - 2A \text{ for } A, \quad 180^\circ - 2B \text{ for } B, \quad 180^\circ - 2C \text{ for } C.$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Divide this equation by the right-hand member,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

It will be seen that there is no limit to the number of relations which can thus be derived.

**Exercises.** — For all triangles, prove

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C,$$

$$\sin^2 A + \sin^2 B + \cos^2 C = 1 + 2 \sin A \sin B \cos C,$$

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C,$$

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

**Exercises.**—The following table contains elements of six different triangles. Assume three independent elements in any one of the triangles and compute other elements.

|                       | 1       | 2       | 3       | 4        | 5       | 6       |
|-----------------------|---------|---------|---------|----------|---------|---------|
| <i>a</i>              | 13      | 25      | 37      | 37       | 68      | 68      |
| <i>b</i>              | 15      | 26      | 13      | 15       | 65      | 75      |
| <i>c</i>              | 14      | 17      | 40      | 44       | 57      | 77      |
| <i>c</i> <sub>1</sub> | 9       | 10      | 5       | 9        | 25      | 45      |
| <i>c</i> <sub>2</sub> | 5       | 7       | 35      | 35       | 32      | 32      |
| <i>s</i>              | 21      | 34      | 45      | 48       | 95      | 110     |
| <i>s</i> − <i>a</i>   | 8       | 9       | 8       | 11       | 27      | 42      |
| <i>s</i> − <i>b</i>   | 6       | 8       | 32      | 33       | 30      | 35      |
| <i>s</i> − <i>c</i>   | 7       | 17      | 5       | 4        | 38      | 33      |
| <i>A</i>              | 53° 8'  | 67° 23' | 67° 23' | 53° 8'   | 67° 23' | 53° 8'  |
| <i>B</i>              | 67° 23' | 73° 44' | 18° 55' | 18° 55'  | 61° 56' | 61° 56' |
| <i>C</i>              | 59° 29' | 38° 53' | 93° 42' | 107° 57' | 50° 41' | 64° 56' |
| <i>K</i>              | 84.00   | 204.0   | 240.0   | 264.0    | 1710    | 2310    |
| <i>R</i>              | 8.125   | 13.54   | 20.04   | 23.12    | 36.83   | 42.50   |
| <i>r</i>              | 4.000   | 6.000   | 5.333   | 5.500    | 18.00   | 21.00   |
| <i>r</i> <sub>1</sub> | 10.50   | 22.67   | 30.00   | 24.00    | 63.33   | 55.00   |
| <i>r</i> <sub>2</sub> | 14.00   | 25.50   | 7.500   | 8.000    | 57.00   | 66.00   |
| <i>r</i> <sub>3</sub> | 12.00   | 12.00   | 48.00   | 66.00    | 45.00   | 70.00   |
| <i>p</i> <sub>1</sub> | 12.92   | 16.32   | 12.97   | 14.27    | 50.29   | 67.94   |
| <i>p</i> <sub>2</sub> | 11.20   | 15.69   | 36.92   | 35.20    | 52.62   | 61.60   |
| <i>p</i> <sub>3</sub> | 12.00   | 24.00   | 12.00   | 12.00    | 60.00   | 60.00   |
| <i>q</i> <sub>1</sub> | 12.95   | 17.11   | 16.33   | 20.01    | 50.53   | 67.96   |
| <i>q</i> <sub>2</sub> | 11.22   | 16.19   | 37.92   | 39.65    | 53.18   | 61.93   |
| <i>q</i> <sub>3</sub> | 12.09   | 24.04   | 13.16   | 12.56    | 60.07   | 60.18   |
| <i>m</i> <sub>1</sub> | 12.97   | 22.49   | 23.29   | 27.17    | 50.80   | 68.09   |
| <i>m</i> <sub>2</sub> | 11.24   | 20.33   | 37.98   | 39.95    | 53.67   | 64.02   |
| <i>m</i> <sub>3</sub> | 12.17   | 24.19   | 19.21   | 17.69    | 60.10   | 61.39   |

## CHAPTER VII.

### SPHERICAL TRIANGLES.

**53.** If there are three points on the surface of a sphere, and each pair of points is joined by an arc of a great circle, the surface of the sphere is divided into two parts, each of which is called a **Spherical Triangle**. Join each of the three points to  $O$ , the center of the sphere. The arcs joining the pairs of points on the surface measure respectively the plane angles formed at the center of the sphere. Thus the

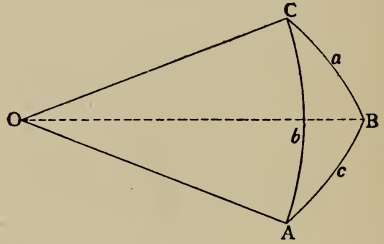


FIG. 64.

arc  $AB$  measures the plane angle  $AOB$ . Also the angles between the arcs are equal respectively to the dihedral angles made by the planes of the arcs. Thus the angle  $A$  of the spherical triangle is equal to the angle between the planes  $ACO$  and  $ABO$ .

There is no limit to the number of different possible values of any side or angle of a spherical triangle. For the arc joining  $A$  and  $B$  may be any arc  $AB +$  any number of circumferences. And the angle  $A$  may be generated by turning  $AC$  away from  $AB$  through any angle  $A +$  any number of revolutions. Since any function of  $n \times 360^\circ + x$  is equal to that function of  $x$ , it follows that any single solution of a spherical triangle includes the solution of all spherical triangles which have any side or angle increased

by any number of times  $360^\circ$ . Therefore, without loss of generality, each side and each angle may be considered less than  $360^\circ$ .

Taking each side less than  $360^\circ$ , there may be:

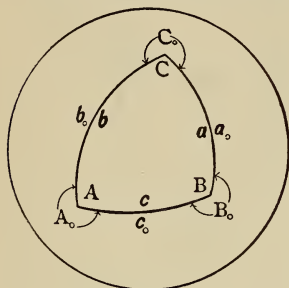


FIG. 65.

$$\begin{aligned} a_0 &= a & A_0 &= 360^\circ - A \\ b_0 &= b & B_0 &= 360^\circ - B \\ c_0 &= c & C_0 &= 360^\circ - C \end{aligned}$$

1. None more than  $180^\circ$ .

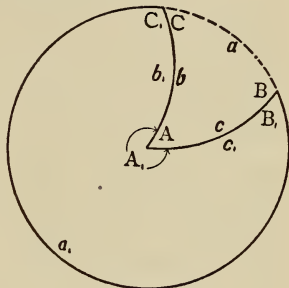


FIG. 66.

$$\begin{aligned} a_1 &= 360^\circ - a & A_1 &= 360^\circ - A \\ b_1 &= b & B_1 &= 180^\circ - B \\ c_1 &= c & C_1 &= 180^\circ - C \end{aligned}$$

2. One more than  $180^\circ$ .

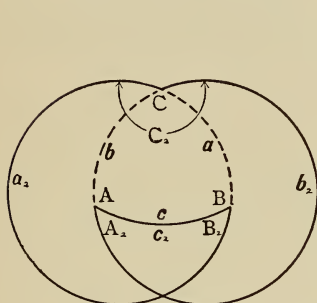


FIG. 67.\*

$$\begin{aligned} a_2 &= 360^\circ - a & A_2 &= 180^\circ - A \\ b_2 &= 360^\circ - b & B_2 &= 180^\circ - B \\ c_2 &= c & C_2 &= 360^\circ - C \end{aligned}$$

3. Two more than  $180^\circ$ .

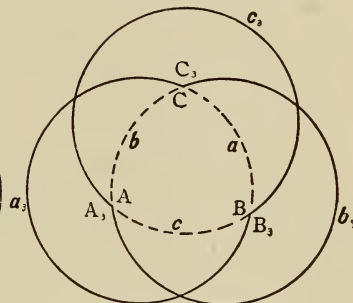


FIG. 68.\*

$$\begin{aligned} a_3 &= 360^\circ - a & A_3 &= A \\ b_3 &= 360^\circ - b & B_3 &= B \\ c_3 &= 360^\circ - c & C_3 &= C \end{aligned}$$

4. Three more than  $180^\circ$ .

\* Out of perspective. It will be well if all the diagrams in this chapter are drawn with colored chalk on a spherical blackboard.

The four triangles in Fig. 69 are called **Associated Triangles**. The three triangles associated with  $ABC$  are obtained by prolonging the sides of  $ABC$ .

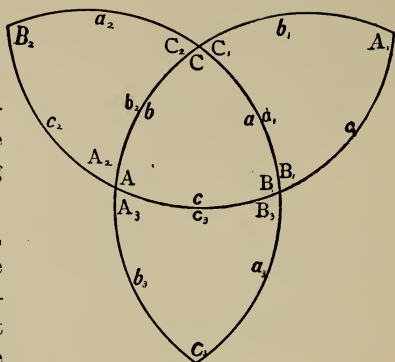


FIG. 69.

Spherical triangles, like plane triangles, are right, isosceles, equilateral, and oblique; but a right spherical triangle may have more than one right angle. If one or more sides of the triangle is a quadrant, the triangle is said to be quadrantal.

#### FORMULÆ FOR RIGHT TRIANGLES.

**54.** Let  $ABC$  be a spherical right triangle, right angled at  $B$ . Let the angles  $A$  and  $C$  be acute, and each side of the triangle be less than  $180^\circ$ . Join  $A$ ,  $B$ , and  $C$  to  $O$ , the center of the sphere. Then the planes  $AOB$  and  $BOC$  are at right angles. From any point  $D$  in the edge  $OC$ , drop a perpendicular  $DB_1$  to the plane  $AOB$ . Through  $DB_1$  pass a plane at right angles to the edge  $OA$ . Since  $DB_1$  is perpendicular to the plane  $AOB$ , it is perpendicular to  $OB_1$  and  $B_1A_1$ ; and since  $A_1B_1$  and  $A_1D$  are both perpendicular to  $OA$ , the angle  $B_1A_1D$  measures the angle between the planes  $AOB$  and  $AOC$ .

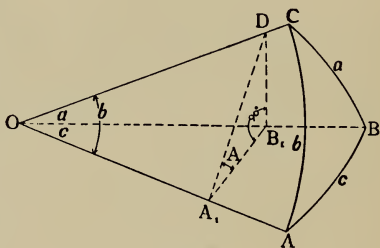


FIG. 70.

$$OD \sin a = B_1D = A_1D \sin A; \text{ and } A_1D = OD \sin b.$$

$$\text{Therefore} \quad \sin a = \sin b \sin A. \quad (35)$$

$$OB_1 \tan a = B_1D = A_1B_1 \tan A; \text{ and } A_1B_1 = OB_1 \sin c.$$

$$\text{Therefore} \quad \tan a = \sin c \tan A. \quad (36)$$

$$OA_1 \tan b = A_1D = A_1B_1 \sec A; \text{ and } A_1B_1 = OA_1 \tan c.$$

$$\text{Therefore} \quad \tan b = \tan c \sec A. \quad (37)$$

If the angle  $A_3B_3C_3$ , Fig. 69, is a right angle, these formulæ (35), (36), and (37) may be shown to apply to the triangle  $A_3B_3C_3$ . Thus put for  $a$  the value  $180^\circ - a_3$ , put for  $b$  the value  $180^\circ - b_3$ , put for  $A$  the value  $180^\circ - A_3$ . There is obtained:

$$\sin (180^\circ - a_3) = \sin (180^\circ - b_3) \sin (180^\circ - A_3);$$

$$\text{whence} \quad \sin a_3 = \sin b_3 \sin A_3.$$

$$\tan (180^\circ - a_3) = \sin c \tan (180^\circ - A_3);$$

$$\text{whence} \quad \tan a_3 = \sin c_3 \tan A_3.$$

$$\tan (180^\circ - b_3) = \tan c \sec (180^\circ - A_3);$$

$$\text{whence} \quad \tan b_3 = \tan c_3 \sec A_3.$$

It therefore follows that the formulæ (35), (36), (37) are true when one of the angles of the right spherical triangle is obtuse. In a similar manner, these formulæ may be shown to apply to the triangle  $A_2B_2C_2$ , Fig. 69, and are therefore true when two of the angles are obtuse.

It therefore follows that formulæ (35), (36), (37) are true for all right triangles in which each of the parts is less than  $180^\circ$ .

In the same three formulæ put successively the values of the parts of the various triangles in Figs. 66 to 68, in which assume  $B = 90^\circ$ . It will be found that all three formulæ apply; they are therefore true for all right spherical triangles.

Collecting,

$$\sin a = \sin b \sin A, \quad (35)$$

$$\tan a = \sin c \tan A, \quad (36)$$

$$\tan b = \tan c \sec A. \quad (37)$$

Changing letters,

$$\sin c = \sin b \sin C, \quad (38)$$

$$\tan c = \sin a \tan C, \quad (39)$$

$$\tan b = \tan a \sec C. \quad (40)$$

$$\text{Sq. root product (35) to (40),} \quad \sec b = \tan A \tan C. \quad (41)$$

$$(41) \div \text{product (36) and (39),} \quad \sec b = \sec a \sec c. \quad (42)$$

$$(35) \times (40) \div (42) \text{ gives} \quad \sec c = \sin A \sec C. \quad (43)$$

$$(37) \times (38) \div (42) \text{ gives} \quad \sec a = \sec A \sin C. \quad (44)$$

**55.** The following diagrams with the accompanying rules will be found useful in rapidly picking out the relation which exists between any three parts of a spherical right triangle.

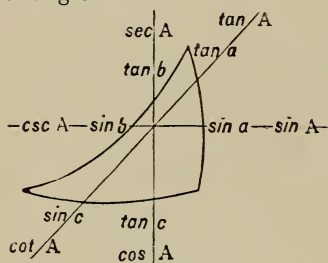


FIG. 71.

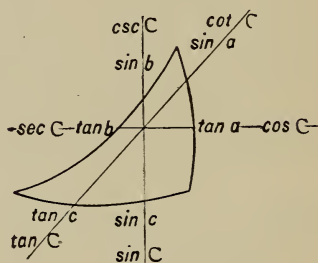


FIG. 72.

**RULE I.** Any function is equal to the product of its adjacent functions.

**RULE II.** Any function is equal to the product of adjacent function and the reciprocal of the next function.

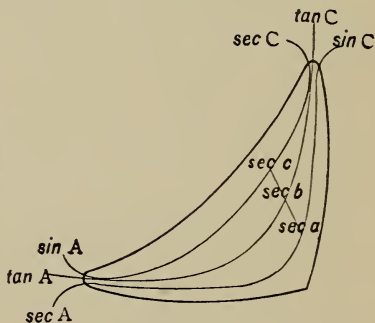


FIG. 73.

**56.** In the discussion of plane right triangles it was found that any two parts, provided one of them was a side, was sufficient to make the solution of a right triangle determinate. In spherical triangles with one right angle, there will always be two solutions (if any) for given values of two given parts; and therefore in addition to values of two parts there must be **included in the data a condition** by which the **quadrant** of any **one of the three unknowns** may be determined. As each unknown is determined by a function, the algebraic sign of any other function (except the reciprocal function) will be sufficient to determine the quadrant of such an unknown.

Given  $a = 86^\circ 40'$ ,  $c = 32^\circ 40'$ ,  $A$  acute; find  $A$  and  $b$ .

$$\tan A = \tan a \csc c \qquad \sec b = \sec a \sec c$$

$$1.2348$$

$$1.2355$$

$$\underline{0.2678}$$

$$\underline{0.0748}$$

$$1.5026$$

$$\underline{1.3103}$$

$$A = 88^\circ 12'$$

$$b = 87^\circ 12'$$

The value of  $b$  cannot be  $272^\circ 48'$ , since the fourth quadrant is inadmissible by either of the two formulæ:

$$\sin b = \sin a \csc A \quad \text{and} \quad \tan b = \tan c \sec A.$$

Had  $A$  been known to be greater than  $90^\circ$ , the required values would have been  $A = 268^\circ 12'$  and  $b = 272^\circ 48'$ .

Given  $A = 213^\circ$ ,  $b = 54^\circ 20'$ , and  $C$  known to be greater than  $180^\circ$ ; find all the parts.

$$\tan C = \sec b \cot A \quad \sin a = \sin b \sin A \quad \tan c = \tan b \cos A$$

$$0.2343$$

$$9.9098$$

$$0.1441$$

$$\underline{0.1875}$$

$$\underline{9.7361}_n$$

$$9.9236_n$$

$$\underline{0.4218}$$

$$\underline{9.6459}_n$$

$$\underline{0.0677}_n$$

$$C = 259^\circ 16'$$

$$a = 333^\circ 44'$$

$$c = 229^\circ 27'$$

Quadrant given by  $\sec a = \sec A \sin C$ ,  $\sec c = \sin A \sec C$ ;  
or  $\tan a = \tan b \cos C$ ,  $\sin c = \sin b \sin C$ .

Given  $b = 53^\circ 12'$ ,  $C = 110^\circ 48'$ ,  $c$  acute; find  $a$  and  $A$ .

$$\tan a = \tan b \cos C$$

$$\tan A = \sec B \cot C$$

$$0.1260$$

$$0.2226$$

$$\underline{9.5504}_n$$

$$\underline{9.5796}_n$$

$$9.6764_n$$

$$9.8022_n$$

$$a = 334^\circ 37'$$

$$A = 327^\circ 37'$$

Given  $A = 327^\circ 37'$ ,  $C = 110^\circ 48'$ ,  $b$  more than  $180^\circ$ .

$$\sec b = \tan A \tan C \quad \sec a = \sec A \sin C \quad \sec c = \sin A \sec C$$

$$9.8022_n$$

$$0.0734$$

$$9.7288_n$$

$$\underline{0.4204}_n$$

$$\underline{9.9707}$$

$$\underline{0.4496}_n$$

$$0.2226$$

$$=$$

$$0.0441$$

$$+$$

$$\underline{0.1784}$$

$$b = 306^\circ 48'$$

$$a = 25^\circ 23'$$

$$c = 311^\circ 32'$$

Given  $A = 42^\circ 20'$ ,  $C = 353^\circ 50'$ . Impossible. Why?

Given  $A = 42^\circ 20'$ ,  $a = 268^\circ$ . Impossible. Why?

Given  $b = 98^\circ$ ,  $a = 268^\circ$ . Impossible. Why?

Assume the values of any two parts of any triangle in any row of the following table and compute the value of any other part in that row:

| $A$             | $C$             | $a$             | $b$             | $c$             |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $61^\circ 35'$  | $30^\circ 29'$  | $20^\circ 17'$  | $23^\circ 13'$  | $11^\circ 32'$  |
| $61^\circ 35'$  | $30^\circ 29'$  | $339^\circ 43'$ | $336^\circ 47'$ | $348^\circ 28'$ |
| $118^\circ 25'$ | $329^\circ 31'$ | $339^\circ 43'$ | $336^\circ 47'$ | $11^\circ 32'$  |
| $298^\circ 25'$ | $149^\circ 31'$ | $339^\circ 43'$ | $23^\circ 13'$  | $11^\circ 32'$  |
| $63^\circ 15'$  | $135^\circ 34'$ | $50^\circ 0'$   | $120^\circ 56'$ | $143^\circ 5'$  |
| $296^\circ 45'$ | $64^\circ 26'$  | $310^\circ 0'$  | $120^\circ 56'$ | $143^\circ 5'$  |
| $116^\circ 45'$ | $224^\circ 26'$ | $310^\circ 0'$  | $239^\circ 4'$  | $143^\circ 5'$  |
| $63^\circ 15'$  | $135^\circ 34'$ | $310^\circ 0'$  | $239^\circ 4'$  | $216^\circ 55'$ |

**57. Polar triangles.**—Let  $ABC$  be a spherical triangle with each part less than  $180^\circ$ . With  $A$ ,  $B$ , and  $C$ , respec-

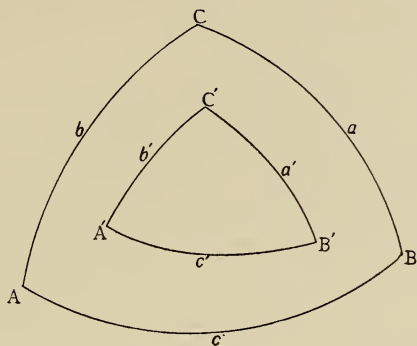


FIG. 74.

tively, as poles, describe great circles  $B'C'$ ,  $C'A'$ , and  $A'B'$ . Now if with  $A'$ ,  $B'$ , and  $C'$  as poles, great circles are described, triangle  $ABC$  is obtained. The two triangles  $ABC$  and  $A'B'C'$ , thus related, are said to be polar to each other. It is proved in geometry that the relations of their parts are:

$$a = 180^\circ - A'; \quad b = 180^\circ - B'; \quad c = 180^\circ - C'.$$

$$A = 180^\circ - a'; \quad B = 180^\circ - b'; \quad C = 180^\circ - c'.$$

**Quadrantal triangles.**—If the triangle  $ABC$  is a right triangle, its polar triangle  $A'B'C'$  is a quadrantal triangle. Hence, substituting in (35), (36), and (37):

$$\sin(180^\circ - A') = \sin(180^\circ - B') \sin(180^\circ - a');$$

$$\text{or} \quad \sin A' = \sin B' \sin a'. \quad (45)$$

$$\text{Similarly,} \quad \tan A' = \sin C' \tan a'; \quad (46)$$

$$\tan B' = -\tan C' \sec a'. \quad (47)$$

These formulæ are true for all quadrantal triangles having each part less than  $180^\circ$ .

Now assume  $b$ , Fig. 66, to be a quadrant.

In (45), (46), and (47), put  $360^\circ - A_1$  for  $A'$ ,  $180^\circ - B_1$  for  $B'$ ,  $180^\circ - C_1$  for  $C'$ , and  $360^\circ - a_1$  for  $a'$ . After reduction there is obtained:  $\sin A_1 = \sin B_1 \sin a_1$ ;  $\tan A_1 = \sin C_1 \tan a$ ;  $\tan B_1 = -\tan C_1 \sec a_1$ .

Hence the formulæ (45), (46), (47), are true for a quadrantal triangle with one angle greater than  $180^\circ$ .

The triangle made up of the hemisphere + the triangle  $ABC$ , Fig. 66, has its angles  $A$ ,  $180^\circ + B$ , and  $180^\circ + C$ , and its sides  $b$ ,  $c$ , and  $360^\circ - a$ . These values also satisfy formulæ (45), (46), (47). The values of the parts of the triangle  $A_0B_0C_0$ , Fig. 65, also satisfy the same formulæ. Hence these formulæ are true for all quadrantal triangles. The formulæ for the solution of a quadrantal triangle are: ( $b = 90^\circ$ ):

$$\begin{aligned} \sin A &= \sin B \sin a, & \sin C &= \sin B \sin c, \\ \tan A &= \sin C \tan a, & \tan C &= \sin A \tan c, \\ \sec A &= \sin c \sec a, & \sec C &= \sin a \sec c, \\ \tan B &= -\tan A \sec c, & \sec B &= -\tan a \tan c, \\ \tan B &= -\tan C \sec a, & \sec B &= -\sec A \sec C. \end{aligned}$$

They are obtainable from the formulæ for right triangles by interchanging the large and small letters and prefixing a minus sign to every function except the sine and cosecant.

**Isosceles triangles** are solved by drawing a perpendicular from the unequal angle upon its opposite side, thus forming two right triangles.

**Regular polygons** are solved by joining the center of the polygon to each vertex and also to the middle of each side, thus forming  $2n$  equal right triangles,  $n$  being the number of sides.

## ELEMENTS OF SOME QUADRANTAL TRIANGLES.

| $a$      | $b$ | $c$      | $A$      | $B$      | $C$      |
|----------|-----|----------|----------|----------|----------|
| 240° 0'  | 90° | 45° 0'   | 225° 0'  | 125° 16' | 144° 44' |
| 103° 26' | 90° | 154° 0'  | 238° 0'  | 240° 41' | 202° 28' |
| 292° 22' | 90° | 48° 50'  | 300° 22' | 68° 55'  | 135° 23' |
| 104° 53' | 90° | 133° 40' | 249° 12' | 255° 18' | 224° 24' |
| 174° 13' | 90° | 94° 8'   | 175° 57' | 135° 34' | 135° 43' |
| 104° 53' | 90° | 133° 40' | 110° 48' | 104° 42' | 135° 36' |
| 299° 47' | 90° | 124° 51' | 307° 15' | 113° 29' | 48° 48'  |

## ELEMENTS OF SOME ISOSCELES TRIANGLES.

| $a$      | $b$      | $c$      | $A$      | $B$      | $C$      |
|----------|----------|----------|----------|----------|----------|
| 54° 20'  | 54° 20'  | 87° 5'   | 47° 0'   | 47° 0'   | 116° 0'  |
| 63° 56'  | 63° 56'  | 302° 0'  | 105° 44' | 105° 44' | 294° 40' |
| 112° 48' | 112° 48' | 254° 9'  | 56° 12'  | 56° 12'  | 240° 8'  |
| 100° 4'  | 100° 4'  | 207° 2'  | 42° 24'  | 42° 24'  | 198° 8'  |
| 127° 50' | 127° 50' | 272° 52' | 42° 24'  | 42° 24'  | 238° 30' |
| 101° 7'  | 101° 7'  | 303° 31' | 83° 57'  | 83° 57'  | 302° 20' |

## ELEMENTS OF SOME REGULAR POLYGONS.

| $n$ | $a$  | $C$      | $n$ | $a$ | $C$      |
|-----|------|----------|-----|-----|----------|
| 3   | 60°  | 70° 32'  | 9   | 20° | 145° 11' |
| 4   | 90°  | 90° 0'   | 10  | 30° | 159° 53' |
| 4   | 60°  | 109° 28' | 12  | 24° | 161° 52' |
| 3   | 108° | 116° 34' | 15  | 20° | 166° 40' |
| 5   | 60°  | 138° 11' | 18  | 18° | 171° 14' |
| 6   | 50°  | 145° 42' | 20  | 12° | 166° 34' |
| 8   | 40°  | 158° 57' | 30  | 10° | 173° 22' |

## OBLIQUE SPHERICAL TRIANGLES.

**58.** As in plane triangles, the solution is effected by dropping a perpendicular from a vertex upon the opposite side. Two right triangles are thus formed, in each of which a value of the perpendicular is found. The solution is then obtained by equating the values of the perpendicular.

This method fails when the given parts are the three sides or the three angles.

The solutions for these two cases will now be found.

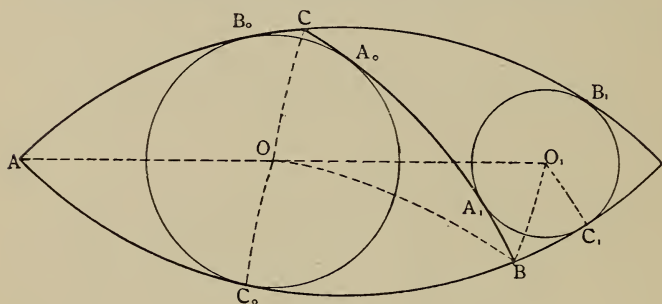


FIG. 75.

**59. The in- and excircles.** — Prolong the sides  $AB$  and  $AC$ , and bisect the angles  $A$ ,  $B$ , and  $CBC_1$  by great circles. From the intersections  $O$  and  $O_1$  drop perpendiculars on  $AB$ . With  $O$  and  $O_1$  as poles, describe small circles with radii  $OC_0$  and  $OC_1$  respectively. It is easy to see that the various triangles  $AOB_0$  and  $AOC_0$ ,  $AO_1B_1$  and  $AO_1C_1$ , etc., are equal in pairs; therefore the small circles touch  $AC$  in  $B_0$  and  $B_1$ , and  $BC$  in  $A_0$  and  $A_1$ .

Then exactly as in plane triangles it may be shown that  $AC_1 = \frac{1}{2}(a + b + c) = s$ ,  $AC_0 = s - a$ ,  $BC_0 = s - b$ ,  $BC_1 = s - c$ , etc.

Since the angles  $ABC + CBC_1 = 180^\circ$ , the sum of their halves, or  $OBO_1 = 90^\circ$ ; therefore,  $O_1BC = 90^\circ - \frac{1}{2}B$ .

In triangle  $AO_1C_1$ ,  $\tan \frac{1}{2} A = \csc s \tan r_1$ .

In triangle  $AOC_0$ ,  $\tan r = \sin (s - a) \tan \frac{1}{2} A$ . (48)

In triangle  $BOC_0$ ,  $\tan r = \sin (s - b) \tan \frac{1}{2} B$ .

In triangle  $BO_1C_1$ ,  $\tan r_1 = \sin (s - c) \cot \frac{1}{2} B$ .

Hence  $\tan r = \sqrt{\csc s \sin (s - a) \sin (s - b) \sin (s - c)}$ . (49)

**60. The incircle of the polar triangle.** — Since (48) is true for all triangles, it is true for the polar of  $ABC$ . That is,

$$\tan r' = \sin (s' - a') \tan \frac{1}{2} A'.$$

Now  $A' = 180^\circ - a$  and  $\frac{1}{2} A' = 90^\circ - \frac{1}{2} a$ .

$2s' = 180^\circ - A + 180^\circ - B + 180^\circ - C$ ; hence  $s' = 270^\circ - S$ , where  $S$  stands for  $\frac{1}{2}(A + B + C)$ ; since  $a' = 180^\circ - A$ ,  $s' - a' = 90^\circ - (S - A)$ . Substituting these values in (48) and taking the reciprocal of the resulting equation,

$$\cot r' = \tan \frac{1}{2} a \sec (S - A). \quad (50)$$

The same process applied to (49) gives

$$\cot r' = \sqrt{-\cos S \sec (S - A) \sec (S - B) \sec (S - C)}.$$

**The circumcircle.** — Bisect the sides of the triangle  $ABC$  by perpendiculars meeting in  $F$ . On account of the equality of the triangles in pairs,  $FA = FB = FC$ , which denote by  $R$ . Also the angles of the isosceles triangles are equal in pairs. Denote  $FBC$  and  $FCB$  by  $x$ ,  $FCA$  and  $FAC$  by  $y$ ,  $FAB$  and  $FBA$  by  $z$ .

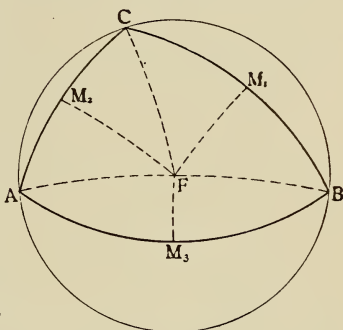


FIG. 76.

Since  $x + y = C$ ,  $y + z = A$ ,  $z + x = B$ ,  $x = S - A$ .

In triangle  $BM_1F$ ,  $\tan R = \tan \frac{1}{2} a \sec (S - A)$ . (51)

By (50),  $\tan R = \cot r'$ . Hence

$$\tan R = \sqrt{-\cos S \sec (S - A) \sec (S - B) \sec (S - C)}. \quad (52)$$

61. Equations (35) to (44) applied to Fig. 77 give

$$\tan c_1 = \tan b \cos A \text{ and } \tan c_2 = \tan a \cos B. \quad (53)$$

$$\tan C_1 = \sec b \cot A \text{ and } \tan C_2 = \sec a \cot B. \quad (54)$$

$$\sin p = \sin b \sin A = \sin a \sin B. \quad (55)$$

$$\sin p = \tan c_1 \cot C_1 = \tan c_2 \cot C_2. \quad (56)$$

$$\tan p = \sin c_1 \tan A = \sin c_2 \tan B. \quad (57)$$

$$\tan p = \tan b \cos C_1 = \tan a \cos C_2. \quad (58)$$

$$\sec p = \sec b \cos c_1 = \sec a \cos c_2. \quad (59)$$

$$\sec p = \sec A \sin C_1 = \sec B \sin C_2. \quad (60)$$

Rearranging (55) and changing letters,

$$\sin a \csc A = \sin b \csc B = \sin c \csc C. \quad (61)$$

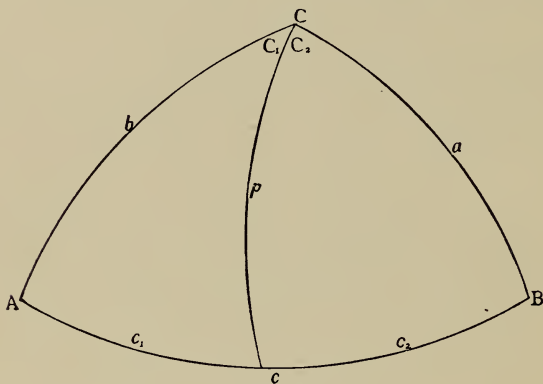


FIG. 77.

62. Methods of solving oblique spherical triangles. — The following methods may be modified in various ways especially to suit problems in which all of the unknown parts are not desired.

(1) Given two sides and the angle opposite one of them, say  $a, b, A$ . Find  $B$  by (55),  $c$  by (53), and  $C$  by (54) or (61).

(2) **Given two angles and the side opposite one of them**, say  $A, B, b$ . Find  $a$  by (55),  $c$  by (53), and  $C$  by (54) or (61).

(3) **Given two sides and the included angle**, say  $b, c, A$ . First find  $c_1$  by (53);  $B$  can be found by (57), and  $a$  by (59). If  $C$  is desired, it can be found by (61). When only one of the unknown angles is desired, draw the perpendicular through the angle which is not desired.

(4) **Given two angles and the included side**, say  $b, A, C$ . First find  $C_1$  by (54);  $a$  can be found by (58),  $B$  by (60),  $c$  by (61). If only one of the unknown sides is desired, draw the perpendicular upon the side which is not desired.

(5) **Given the three sides**.—The angles can be found by (49), (48), etc.

(6) **Given the three angles**.—The sides can be found by (52), (51), etc.

## SOLUTIONS.

**63. 1. Given two sides and the angle opposite one of them**,  $a = 236^\circ 5'2$ ,  $b = 93^\circ 40'$ ,  $A = 220^\circ$ , and  $B$  known to be between  $90^\circ$  and  $180^\circ$ ; find  $c$  and  $B$ .

$$\tan c_1 = \tan b \cos A$$

$$\cos c_2 = \sec b \cos c_1 \cos a$$

$$\log \tan b \quad 1.1933 \, n$$

$$\log \sec b \quad 1.1941 \, n$$

$$L \cos A \quad \underline{9.8843 \, n}$$

$$L \cos c_1 \quad 8.9211$$

$$\log \tan c_1 \quad \underline{1.0776}$$

$$L \cos a \quad \underline{9.7377 \, n}$$

$$c_1 = 85^\circ 13', \quad c_2 = 315^\circ 27'.$$

$$L \cos c_2 \quad \underline{9.8529}$$

$$\sin B = \sin A \csc a \sin b$$

$$\text{Check: } \tan c_2 = \tan a \cos B$$

$$L \sin A \quad 9.8081 \, n$$

$$L \tan c_2 \quad 9.9932 \, n$$

$$\log \csc a \quad 0.0771 \, n$$

$$\log \tan a \quad 0.1853$$

$$L \sin b \quad \underline{9.9991}$$

$$L \cos B \quad 9.8081 \, n$$

$$L \sin B \quad \underline{9.8843}$$

$$c = 40^\circ 40', \quad B = 130^\circ.$$

2. Given two angles and the side opposite one of them,  $A = 130^\circ$ ,  $B = 110^\circ$ ,  $b = 126^\circ 58'$ ; find  $C$  (less than  $90^\circ$ ).

$$\begin{array}{ll} \tan C_1 = \sec b \cot A & \sin C_2 = \sec A \sin C_1 \cos B \\ \log \sec b & 0.2209 \, n \quad \log \sec A \quad 0.1919 \\ L \cot A & 9.9238 \quad L \sin C_1 \quad 9.9100 \\ \log \tan C_1 & \underline{0.1447} \quad L \cos B \quad \underline{9.5341} \\ C_1 = 54^\circ 22' & L \sin C_2 \quad \underline{9.6360} \end{array}$$

$C_2 = 25^\circ 38'$ , and hence  $C = 80^\circ$ .

3. Given two sides and the included angle,  $b = 251^\circ 30'$ ,  $c = 275^\circ 10'$ ,  $A = 132^\circ 30'$ , and  $a$  less than  $180^\circ$ ; find  $a$  and  $B$ .

$$\begin{array}{lll} \tan c_1 = \tan b \cos A; & c_2 = c - c_1; \\ \sec a = \sec b \cos c_1 \sec c_2; & \tan B = \tan A \sin c_1 \csc c_2. \\ \log \tan b & 0.4755 & \log \sec b \quad 0.4985 \, n \quad \log \tan A \quad 0.0379 \\ L \cos A & \underline{9.8297} \, n & L \cos c_1 \quad 9.6470 \, n \quad L \sin c_1 \quad 9.9524 \\ \log \tan c_1 & \underline{0.3052} \, n & \log \sec c_2 \quad \underline{0.0303} \, n \quad \log \csc c_2 \quad \underline{0.4424} \\ c_1 = 116^\circ 20' & \log \sec a & \underline{0.1758} \, n \quad \log \tan B \quad \underline{0.4327} \\ c_2 = 158^\circ 50' & a = 131^\circ 50' & B = 290^\circ 16' \end{array}$$

4. Given two angles and the included side,  $b = 251^\circ 30'$ ,  $A = 132^\circ 30'$ ,  $C = 279^\circ 50'$ , and  $a$  less than  $180^\circ$ ; find  $a$  and  $B$ .

$$\begin{array}{lll} \tan C_1 = \sec b \cot A; & C_2 = C - C_1; \\ \tan a = \tan b \cos C_1 \sec C_2; & \sec B = \sec A \sin C_1 \csc C_2. \\ \log \sec b & 0.4985 \, n & \log \tan b \quad 0.4755 \quad \log \sec A \quad 0.1703 \, n \\ L \cot A & \underline{9.9621} \, n & L \cos C_1 \quad 9.5148 \, n \quad L \sin C_1 \quad 9.9754 \, n \\ \log \tan C_1 & \underline{0.4606} & \log \sec C_2 \quad \underline{0.0579} \quad \log \csc C_2 \quad \underline{0.3153} \\ C_1 = 250^\circ 54' & \log \tan a & \underline{0.0482} \, n \quad \log \sec B \quad \underline{0.4610} \\ C_2 = 28^\circ 56' & a = 131^\circ 50' & B = 290^\circ 15' \end{array}$$

5. Given the three sides,  $a = 124^\circ$ ,  $b = 54^\circ$ ,  $c = 98^\circ$ , and  $A$  less than  $180^\circ$ ; find the angles.

$$\tan r = \sqrt{\csc s \sin(s-a) \sin(s-b) \sin(s-c)};$$

$$\cot \frac{1}{2} A = \frac{\sin(s-a)}{\tan r}; \cot \frac{1}{2} B = \frac{\sin(s-b)}{\tan r}; \text{ etc.}$$

6. Given the three angles,  $A = 220^\circ$ ,  $B = 130^\circ$ ,  $C = 150^\circ$ , and  $a$  known to be more than  $180^\circ$ ; find the sides.

$$\tan R = \sqrt{-\cos S \sec(S-A) \sec(S-B) \sec(S-C)};$$

$$\cot \frac{1}{2} a = \frac{\sec(S-A)}{\tan R}; \text{ etc.} \quad \text{Check by (61).}$$

*Computation for Problem 5.*

|                           |        |
|---------------------------|--------|
| $\log \csc 138^\circ$     | 0.1745 |
| $L \sin 14^\circ$         | 9.3837 |
| $L \sin 84^\circ$         | 9.9976 |
| $L \sin 40^\circ$         | 9.8081 |
| $L \tan^2 r$              | 9.3639 |
| $L \tan r$                | 9.6819 |
| $L \cot \frac{1}{2} A$    | 9.7018 |
| $\log \cot \frac{1}{2} B$ | 0.2957 |
| $\log \cot \frac{1}{2} C$ | 0.1062 |

$$A = 126^\circ 34', \quad B = 53^\circ 42', \\ \text{and } C = 76^\circ 6'.$$

If  $A$  is known to be greater than  $180^\circ$ , the negative root of  $\tan^2 r$  would have been taken. Hence we would find

$$A = 233^\circ 26', \quad B = 306^\circ 18', \\ C = 283^\circ 54'. \quad (\text{See Fig. 65.})$$

As a check, apply (61).

*Computation for Problem 6.*

|                           |            |
|---------------------------|------------|
| $L(-\cos 250^\circ)$      | 9.5341     |
| $\log \sec 30^\circ$      | 0.0625     |
| $\log \sec 120^\circ$     | 0.3010 $n$ |
| $\log \sec 100^\circ$     | 0.7603 $n$ |
| $\log \tan^2 R$           | 0.6579     |
| $\log \tan R$             | 0.3289 $n$ |
| $L \cot \frac{1}{2} a$    | 9.7336 $n$ |
| $L \cot \frac{1}{2} b$    | 9.9721     |
| $\log \cot \frac{1}{2} c$ | 0.4314     |

$$a = 236^\circ 52', \quad b = 93^\circ 40', \\ \text{and } c = 40^\circ 38'.$$

|               |            |
|---------------|------------|
| $L \sin a$    | 9.9229 $n$ |
| $\log \csc A$ | 0.1919 $n$ |
| $L \sin b$    | 9.9991     |
| $\log \csc B$ | 0.1157     |
| $L \sin c$    | 9.8137     |
| $\log \csc C$ | 0.3010     |
| Check:        | 0.1148     |

**64.** It is frequently convenient to express *a desired part directly in terms of given parts*, especially when a chain of triangles is to be solved. It is recommended that the following formulæ be derived from a diagram, in the manner that the formulæ for right triangles were derived.

They will here be obtained by applying equations (67) and (68) proved in Appendix II., page 145.

Thus, from (59), (68), and (53),

$$\begin{aligned}\cos a &= \cos b \sec c_1 \cos (c - c_1) \\ &= \cos b \sec c_1 (\cos c \cos c_1 + \sin c \sin c_1) \\ &= \cos b \cos c + \cos b \sin c \tan c_1 \\ &= \cos b \cos c + \cos b \sin c \tan b \cos A; \\ \cos a &= \cos b \cos c + \sin b \sin c \cos A.\end{aligned}$$

Multiply by  $\csc b \csc c$ , and transpose :

$$\cos A = -\cot b \cot c + \csc b \csc c \cos a.$$

Apply the last two equations to the polar triangle :

$$\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos a &= \cot B \cot C + \csc B \csc C \cos A.\end{aligned}$$

Using in the same manner (57), (67), and (53),

$$\begin{aligned}\cot B &= -\cos c \cot A + \sin c \csc A \cot b, \\ \cot b &= \cot c \cos A + \csc c \sin A \cot B.\end{aligned}$$

By changing letters, similar relations between other parts of the triangle may be obtained.

The application of these forms involves the use of both tables,—the table of logarithms of functions in computing the values of each term of the right-hand member in any equation and the table of natural values of functions in finding the answer.

As an illustration of the application of these formulæ, find  $a$  (less than  $180^\circ$ ) given

$$b = 127^\circ, \quad c = 57^\circ, \quad A = 130^\circ.$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

|                        |                     |            |                        |
|------------------------|---------------------|------------|------------------------|
|                        |                     | $L \cos b$ | 9.7795 <i>n</i>        |
|                        |                     | $L \cos c$ | 9.7361                 |
|                        |                     |            | <u>9.5156 <i>n</i></u> |
| $\cos b \cos c$        | $= - .3278$         | $L \sin b$ | 9.9023                 |
| $\sin b \sin c \cos A$ | $= - .4305$         | $L \sin c$ | 9.9236                 |
| nat. $\cos a$          | $= - .7583$         | $L \cos A$ | 9.8081 <i>n</i>        |
| Therefore              | $a = 139^\circ 19'$ |            | <u>9.6340 <i>n</i></u> |

With the same data, find  $B$  (also less than  $180^\circ$ ).

$$\cot B = -\cos c \cot A + \sin c \csc A \cot b.$$

|                     |               |                        |
|---------------------|---------------|------------------------|
|                     | $L(-\cos c)$  | 9.7361 <i>n</i>        |
|                     | $L \cot A$    | 9.9238 <i>n</i>        |
|                     |               | <u>9.6599</u>          |
| .4570               | $L \sin c$    | 9.9236                 |
| $- .8249$           | $\log \csc A$ | 0.1157                 |
| $- .3679$           | $L \cot b$    | 9.8771 <i>n</i>        |
| $B = 110^\circ 12'$ |               | <u>9.9164 <i>n</i></u> |

## ELEMENTS OF SOME OBLIQUE SPHERICAL TRIANGLES.

| $a$      | $b$      | $c$      | $A$      | $B$      | $C$      |
|----------|----------|----------|----------|----------|----------|
| 131° 35' | 108° 30' | 84° 47'  | 132° 14' | 110° 11' | 99° 42'  |
| 124° 13' | 125° 42' | 82° 48'  | 127° 22' | 128° 42' | 107° 33' |
| 134° 16' | 150° 57' | 55° 42'  | 120° 48' | 144° 23' | 97° 43'  |
| 126° 25' | 138° 32' | 45° 54'  | 261° 16' | 234° 25' | 298° 6'  |
| 309° 48' | 116° 45' | 129° 12' | 59° 4'   | 274° 23' | 300° 5'  |
| 120° 56' | 59° 4'   | 106° 10' | 116° 45' | 63° 15'  | 91° 7'   |
| 113° 4'  | 111° 41' | 79° 35'  | 110° 50' | 109° 17' | 87° 35'  |
| 126° 0'  | 152° 0'  | 75° 0'   | 142° 24' | 159° 16' | 133° 14' |
| 90° 53'  | 117° 49' | 132° 5'  | 120° 0'  | 130° 0'  | 140° 0'  |
| 155° 5'  | 147° 6'  | 33° 2'   | 133° 18' | 110° 10' | 70° 21'  |

## THE CELESTIAL SPHERE.

**65.** The concave surface on which the heavenly bodies seem to be situated is called the celestial sphere.

**Intersections of the celestial sphere :**

I. Its intersection with the plane of the earth's equator is called the **celestial equator**.

II. Its intersections with the prolongations of the earth's axis are called the **poles** of the heavens.

III. Its intersection with a plane tangent to the earth through the foot of an observer is called the **horizon** of the observer.

IV. Its intersections with the vertical line through an observer are called the **zenith** and **nadir** respectively.

Great circles through the poles are called **declination circles** (or **hour circles**). On account of the motion of the earth on its axis, these circles seem to move from east to west  $15^\circ$  every hour.

Great circles through the zenith and nadir of an observer are called **vertical circles**. The vertical circle through the east and west points of the horizon is called the **prime vertical** of the observer. The vertical circle through the north and south points passes also through the poles ; it is called the **meridian** of the observer.

The points of intersection of the celestial equator with the path of the sun are called **equinoxes**. The vernal equinox is passed by the sun on March 21, and the autumnal equinox is passed on September 21.

The arc of the celestial equator from the vernal equinox to the declination circle through a heavenly body, say a star, is called the **right ascension** of the star. The right ascension is usually expressed in terms of time ( $15^\circ=1$  hour) and is measured toward the east. The right ascension of the

meridian at any time is given by a sidereal clock set for that meridian.

The angle which the declination circle makes with the meridian (= arc on the equator from meridian to declination circle) is called the **hour-angle** of the star. The arc of the declination circle from the celestial equator to the star is called the **declination** of the star. The declination is positive when measured from the equator to the north.

The arc of the horizon from its south point to the vertical circle through a star is called the **azimuth** of the star. The arc of the vertical circle from the horizon to the star is called the **altitude** of the star. The azimuth is positive when measured in the direction SWNE, and the altitude is positive when measured toward the zenith.

These definitions are illustrated in the following diagram.

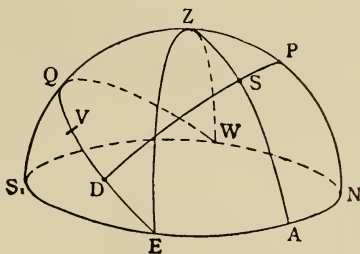


FIG. 78.

|   |         |           |                |          |                 |
|---|---------|-----------|----------------|----------|-----------------|
| N | north   | QZ        | latitude       | VD       | right ascension |
| P | pole    | EQW       | equator        | QPD      | hour angle      |
| Z | zenith  | $S_1WNE$  | horizon        | DS       | declination     |
| V | equinox | EZW       | prime vertical | $S_1WNA$ | azimuth         |
| S | star    | NPZ $S_1$ | meridian       | AS       | altitude        |

The triangle  $PSZ$  is sometimes called the **Astronomical Triangle**. Its parts are:

Arc  $PZ$  = arc  $QP$  - arc  $QZ$  =  $90^\circ$  - latitude.

Arc  $ZS$  = arc  $AZ$  - arc  $AS$  =  $90^\circ$  - altitude.

Arc  $SP$  = arc  $DP$  - arc  $DS$  =  $90^\circ$  - declination.

Angle  $PZS = \text{arc } S_1 WNA - \text{arc } S_1 WN = \text{azimuth} - 180^\circ$ .

Angle  $ZSP$  is called the parallactic angle of the star.

Angle  $ZPS = \text{arc } QD = \text{hour-angle}$ .

**Notation.**—The hour-angle will be denoted by  $P$ , the declination by  $d$ , the azimuth by  $Z$ , the altitude by  $a$ , the latitude by  $l$ .

#### APPLICATIONS TO GEOGRAPHY.

**115.** Find in geographical miles ( $60 \text{ miles} = 1^\circ$ ) the shortest distance on the surface of the earth from Eastport ( $10^\circ \text{ E. longitude, } 44^\circ 50' \text{ N. latitude}$ ) to Point Barrow ( $80^\circ \text{ W. longitude, } 71^\circ 20' \text{ N. latitude}$ ).

**116.** Find the shortest distance on the surface of the earth from Quito ( $2^\circ \text{ W. longitude, } 0^\circ \text{ latitude}$ ) to Point Barrow.

**117.** A required point is described as being equidistant from Eastport, Point Barrow, and Quito. What is the distance on the surface of the earth from the point to each of the three places?

**118.** Find the distance from Eastport to San Diego ( $40^\circ \text{ W., } 32^\circ 40' \text{ N.}$ ).

**119.** The latitude of Key West is  $24^\circ 10' \text{ N.}$ ; the latitude of Attou Island is  $53^\circ \text{ N.}$  The distance apart of the two places is 3722 geographical miles. Find their difference in time.

**120.** A ship sails from Halifax ( $\text{lat. } 44^\circ 40' \text{ N., long. } 13^\circ 25' \text{ E.}$ ) in a direction  $\text{N. } 81^\circ \text{ E.}$ , and continues on an arc of a great circle for 2472 miles. Find the place arrived at.

#### APPLICATIONS TO ASTRONOMY.

**121.** Given the latitude of a place and the declination of the sun, to find the azimuth of the sun at sunrise and the length of the day.

Consider the following cases :

- i.  $l = 40^\circ$ ,  $d = + 23^\circ 27'$ ; the longest day.
- ii.  $l = 40^\circ$ ,  $d = 0^\circ$ ; days and nights equal.
- iii.  $l = 40^\circ$ ,  $d = - 23^\circ 27'$ ; the shortest day.
- iv.  $l = 66^\circ 33'$ ,  $d = 23^\circ 27'$ ; or  $l$  and  $d$  complementary.

**122.** Given the latitude of a place and the declination of the sun, to find when the sun is due west (in the prime vertical). Consider the following cases :

- i.  $l = 40^\circ$ ,  $d = + 23^\circ 27'$ .
- ii.  $l = 40^\circ$ ,  $d = 0^\circ$ .
- iii.  $l = 66^\circ 33'$ ,  $d = + 23^\circ 27'$ .

**123.** Given the latitude of a place and the declination of the sun, to find the altitude and azimuth of the sun at 6 o'clock in the morning. Take  $l = 40^\circ$ ,  $d = 23^\circ 27'$ .

**124.** The interval which elapsed between the transits of Regulus over the prime vertical was observed to be 4 hours and 12 minutes. Find the latitude of the place, the declination of Regulus being  $12^\circ 40'$ .

**125.** At about 8 o'clock, the altitude of Regulus was observed to be  $16^\circ 40'$ . Three hours and twenty-four minutes later, it was again observed and found to be  $40^\circ 20'$ . Find the latitude of the place.

**126.** The interval between the observed equal altitudes of Antares was 4 hours and 20 minutes. Find the latitude of the place if the observed altitudes were  $54^\circ 20'$  and the declination of Antares is  $- 26^\circ 6'$ .

**127.** From a place north of the equator, the altitude of Capella was observed to be  $76^\circ 40'$  when its hour-angle was 1 hour and 6 minutes, the declination of Capella being  $45^\circ 54'$ . Find the latitude of the place of observation.

A tree 100 feet high stands on a horizontal plane in  $40^\circ$  north latitude. What is the area described by its shadow between the hours of 9 and 2 on the 21st of March?

Let  $h$  be the height of the tree,  $s$  the length of its shadow,  $Z$  the azimuth of the sun and of the shadow,  $P$  the hour-angle,  $l$  the latitude, and  $a$  the altitude of the sun.

*First*,  $\cos Z = -\tan a \tan l$ , and by plane trigonometry,  $s \tan a = h$ . The product of these is  $s \cos Z = -h \tan l$ . But  $-s \cos Z$  is the northerly projection of the shadow, and  $h \tan l$  is a constant. Hence, on this day the amount which the shadow extends north is a constant. Therefore the path of the shadow of the top of the tree is a straight line, due east and west, 83.91 feet from the foot of the tree.

*Second*,  $\cot Z = \sin l \cot P$ , from which the azimuth of the shadow at 9 o'clock is  $237^\circ 16'$  and at 2 o'clock it is  $138^\circ 4'$ .

The area of the triangle passed over by the shadow between 9 and 12 is 5477 square feet, and the area described between 12 and 2 is 3162 square feet, making the total area 8639 square feet.

#### AREA OF A SPHERICAL TRIANGLE.

**66.** Denote the area by  $K$ , the half sum of the angles by  $S$ , the radius of the sphere by  $R$ . Then, by geometry,

$$K = \frac{S - 90^\circ}{90^\circ} \pi R^2.$$

Thus, if  $A = 130^\circ$ ,  $B = 110^\circ$ ,  $C = 80^\circ$ ;  $K = 2.4435 R^2$ .

Also, if  $A = 220^\circ$ ,  $B = 130^\circ$ ,  $C = 150^\circ$ ;  $K = 5.5851 R^2$ .

**128.** Given  $b = 70^\circ 20'$ ,  $c = 38^\circ 28'$ ,  $A = 52^\circ 30'$ ; find  $K$ .

**129.** Given  $a = 139^\circ 21'$ ,  $b = 126^\circ 58'$ ,  $c = 56^\circ 52'$ ; find  $K$ .

# APPENDIX I.

## THEORY OF LOGARITHMS.

**67. Arithmetic and geometric series.**—Logarithms are a series of numbers severally assigned to ordinary numbers, and so chosen that the multiplication and division of ordinary numbers may be accomplished by the addition or subtraction of their logarithms.

Thus, if the ordinary numbers are terms in a geometric progression (a multiplication series) and to each term we assign a term in an arithmetic progression (an addition series), it can be shown that the multiplication or division of the terms in the geometric progression can be accomplished by the addition or subtraction of terms in the arithmetic progression.

| NUMBERS. | LOGS.     | Thus suppose it is desired to multiply the numbers $r^5(32)$ and $r^3(8)$ . The logarithm of $r^5(32)$ is $a + 5d(5)$ , and the logarithm of $r^3(8)$ is $a + 7d(7)$ . Adding these logarithms gives $2a + 12d(12)$ , which is larger than the logarithm of the | NUMBERS. | LOGS. |
|----------|-----------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|-------|
| $r^{10}$ | $a$       |                                                                                                                                                                                                                                                                 | 1024     | 0     |
| $r^9$    | $a + d$   |                                                                                                                                                                                                                                                                 | 512      | 1     |
| $r^8$    | $a + 2d$  |                                                                                                                                                                                                                                                                 | 256      | 2     |
| $r^7$    | $a + 3d$  |                                                                                                                                                                                                                                                                 | 128      | 3     |
| $r^6$    | $a + 4d$  |                                                                                                                                                                                                                                                                 | 64       | 4     |
| $r^5$    | $a + 5d$  |                                                                                                                                                                                                                                                                 | 32       | 5     |
| $r^4$    | $a + 6d$  |                                                                                                                                                                                                                                                                 | 16       | 6     |
| $r^3$    | $a + 7d$  |                                                                                                                                                                                                                                                                 | 8        | 7     |
| $r^2$    | $a + 8d$  |                                                                                                                                                                                                                                                                 | 4        | 8     |
| $r$      | $a + 9d$  |                                                                                                                                                                                                                                                                 | 2        | 9     |
| 1        | $a + 10d$ |                                                                                                                                                                                                                                                                 | 1        | 10    |

correct product  $[r^8(256)]$  by  $a + 10d(10)$ .

But  $a + 10d(10)$  is the logarithm of 1.

$$\text{Hence } \log(r^5 \times r^3) = \log r^5 + \log r^3 - \log 1,$$

$$\log(32 \times 8) = \log 32 + \log 8 - \log 1.$$

And this relation is perfectly general in any two such series. Either by trial or reduction from this relation we may obtain

$$\log(pq) = \log p + \log q - \log 1,$$

$$\log \frac{p}{q} = \log p - \log q + \log 1,$$

$$\log p^n = n \log p - (n - 1) \log 1,$$

$$\log \sqrt[n]{p} = \frac{1}{n}(\log p + \log 1).$$

It will be seen that a logarithm is a function fulfilling the condition

**Function  $(ab) = \text{function } (a) + \text{function } (b) - \text{a constant.}$**

The theory of logarithms is principally devoted to obtaining the logarithm of any number whatever, — in other words, interpolating in the two series.

Logarithms may be considered geometrically, as they were by the inventor of the first system:

#### NAPIERIAN LOGARITHMS.\*

**68.** Suppose two points,  $A$  and  $B$ , to start at the same time with the same initial velocity.  $A$  starts at  $A_0$  and moves along the half-line  $A_0A_x$  with a uniform velocity.  $B$  starts at  $B_0$  and

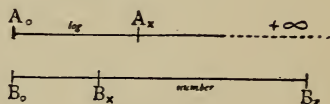


FIG. 79.

\* This name has by some writers been incorrectly applied to Speidell's Logarithms, which were published in 1619, five years after the publication of Napier's logarithmic tables.

moves along the segment  $B_0B_z$ , with a changing velocity such that, when  $B$  arrives at any point  $B_x$ , the velocity of  $B$  will be proportional to the remaining segment  $B_xB_z$ . Then, if at any time  $A$  is at  $A_x$  and at the same time  $B$  is at  $B_x$ , the length of  $A_0A_x$  was called by Napier the logarithm of  $B_xB_z$ .\*

The numerical value of the length of  $B_0B_z$  was taken by Napier to be  $10^7$ , or 10,000,000. It will here be denoted by  $k$ ; hence from the definition, Napierian  $\log k = \text{zero}$ .

Suppose that during the first unit of time  $A$  moves a distance 1, and  $B$  moves a distance which denote by  $(1-r)k$ .† The distance of the moving point  $B$  from  $B_z$  will then be  $k - (1-r)k = rk$ ; and hence Napierian  $\log rk = 1$ .

Now during the second unit of time  $A$  will move another distance equal to 1, making its distance from  $A_0$  equal to 2. Since the law of the diminution of the velocity of  $B$  remains the same throughout its motion, we obtain the distance traversed by it in the second unit of time in the manner as it was obtained for the first unit of time. That is, multiply the distance which it is away from  $B_z$  at the beginning of the unit of time under consideration by the constant  $1-r$ . This gives for the space traveled over by  $B$  in the second unit of time  $(1-r)rk$ ; so that  $B$  is then removed from  $B_z$  by the distance  $rk - (1-r)rk = r^2k$ .

Hence                      Napierian  $\log r^2k = 2$ .

Similarly,              Napierian  $\log r^3k = 3$ ;

and in general,        Napierian  $\log r^nk = n$ .

\* *Mirifici Logarithmorum Canonis Constructio*. English translation by W. R. Macdonald, Edinburgh, 1889.

† If the initial velocities of  $A$  and  $B$  are equal as assumed by Napier, and  $k$  is very large, the value of  $r$  will be nearly equal, but slightly less than 1. The distance gone by  $B$  in the first unit of time will be slightly less than 1 on account of the retardation of the velocity of  $B$  as it approaches  $B_z$ .

It will be seen that the numbers form a geometric progression and their logarithms form an arithmetic progression, and hence that Napierian logarithms are only a particular case of the more general logarithms. Napier solved the problem of finding corresponding intermediate terms in the two series; his logarithms are no longer used, but the method by which we could interpolate in the Napierian series is quite similar to the method adopted for the same process applied to systems now in use.

### PRESENT SYSTEM OF LOGARITHMS.

69. Suppose two points  $A$  and  $B$ , moving in the same direction,  $A$  with a uniform velocity along the line  $A_0A_x$ , and  $B$  along the half-line  $B_0B_x$ , with a velocity directly proportional to its distance

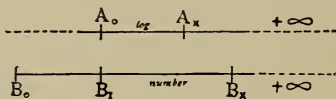


FIG. 80.

from  $B_0$ . If at any time the distance of  $A$  from  $A_0$  is  $A_0A_x$ , and at the same time the distance of  $B$  from  $B_0$  is  $B_0B_x$ , the length  $A_0A_x$  is defined as the logarithm of  $B_0B_x$ .

If it is assumed that  $A$  is at zero ( $A_0$ ) when  $B$  is at 1 ( $B_1$ ), the logarithm of  $1 = 0$ . This assumption avoids the necessity of subtracting the logarithm of 1 when multiplying by means of logarithms.

Denote the velocity which  $B$  has acquired when at  $B_1$  by  $v$ ; then at any other point the velocity of  $B$  will be  $v$  times its distance from  $B_0$ ; thus, the velocity of  $B$  at any point  $B_x = v$  times  $B_0B_x$ . In the first unit after  $B$  leaves  $B_x$ ,  $A$  will be supposed to move a distance 1, and  $B$  will move in the same time a distance, which may be denoted by  $cv$  times  $B_0B_x$ ; for a particular system of logarithms,  $c$  and  $v$  are constants.

Let us count from the instant that  $A$  was at  $A_0$  and  $B$  at  $B_1$ . At this instant  $B_x$  will coincide with  $B_1$ , and the dis-

tance of  $B$  from  $B_0 = 1$ . In the first unit of time afterward, the distance moved over by  $B$  becomes  $cv$ , and its total distance from  $B_0$  at the end of this first unit of time is  $1 + cv$ ; therefore,  $\log(1 + cv) = 1$ .

Under the same assumptions, in the second unit of time  $A$  moves another unit. This makes the distance  $A_0 A_x = 2$ , and the motion of  $B = cv$  times its distance from  $B_0$  at the beginning of this unit of time  $= cv(1 + cv)$ . Therefore, at the end of the unit of time, the distance of  $B$  from  $B_0$

$$= 1 + cv + cv(1 + cv) = (1 + cv)^2.$$

Therefore  $\log(1 + cv)^2 = 2$ .

By a similar process of reasoning it may be shown that  $\log(1 + cv)^3 = 3$ , and in general that  $\log(1 + cv)^n = n$ .

**70.** It will be seen that the values of the distances of  $A$  from  $A_0$  at the ends of successive units of time; namely, 1, 2, 3 ...  $n$ , form an arithmetic progression: and that the values of the distances of  $B$  from  $B_0$ ; namely,  $(1 + cv)$ ,  $(1 + cv)^2$ ,  $(1 + cv)^3$  ...  $(1 + cv)^n$ , form a geometric progression. The value of  $c$  depends on the value assumed for  $v$ , but  $v$  is perfectly arbitrary, and values for it may be assumed at pleasure; now a system of logarithms may be constructed by assuming for  $v$  any value at all, since it has already been shown that the terms of any arithmetic progression may be used as logarithms of the terms of any geometric progression. But in systems of logarithms in actual use, special values have been given to  $v$  on account of special conveniences served thereby. Thus, in higher mathematics, the reciprocal of  $v$  arises as a factor; this factor (denoted by  $M$ ) is eliminated and processes simplified by making  $M = 1$  (whence also  $v = 1$ ). The system in which  $M$  and  $v$  are taken as unity is called the **Natural System of Logarithms**, and the logarithms in that system are called **Natural Logarithms**, or Speidell's logarithms.

**71. The number corresponding to a given logarithm.**

Let us now imagine :

(1) That  $B$ , having an initial velocity of  $v$ , moves for one unit of time divided into  $n$  parts.

(2) That  $B$ , having an initial velocity of 1, moves for  $v$  units of time, each unit being divided into  $n$  parts.

In each case denote the value of the constant, which is  $c$  for one unit of time, by  $\frac{1}{m}$  for each  $\frac{1}{n}$  of the unit of time.

Then under the first assumption the motion of  $B$  is  $\left(1 + \frac{v}{m}\right)^n$ ; and under the second assumption it is  $\left(1 + \frac{1}{m}\right)^{nv}$ .

By the binomial formula,

$$\left(1 + \frac{v}{m}\right)^n = 1 + \frac{nv}{m} + \frac{n(n-1)v^2}{1 \cdot 2 \cdot m^2} + \frac{n(n-1)(n-2)v^3}{1 \cdot 2 \cdot 3 \cdot m^3} + \text{etc.}$$

$$\left(1 + \frac{1}{m}\right)^{nv} = 1 + \frac{nv}{m} + \frac{n^2v^2 - nv}{1 \cdot 2 \cdot m^2} + \frac{n^3v^3 - 3n^2v^2 + 2nv}{1 \cdot 2 \cdot 3 \cdot m^3} + \text{etc.}$$

Now when  $n$  is very great, the distance that  $B$ , with a velocity  $v$  (or 1 under the second assumption) will go from  $B_1$  in  $\frac{1}{n}$  of a unit of time will approach as a limit  $\frac{v}{n}$  (or  $\frac{1}{n}$ ). But this distance is also  $v$  (or 1) times  $\frac{1}{m}$ . Therefore, when  $n$  is infinite,  $m = n$ . Also, as  $m$  and  $n$  become very great,

$$\frac{n(n-1)v^2}{1 \cdot 2 \cdot m^2} \text{ and } \frac{n^2v^2 - nv}{1 \cdot 2 \cdot m^2} \text{ both approach } \frac{n^2v^2}{1 \cdot 2 \cdot m^2};$$

$$\frac{n(n-1)(n-2)v^3}{1 \cdot 2 \cdot 3 \cdot m^3} \text{ and } \frac{n^3v^3 - 3n^2v^2 + 2nv}{1 \cdot 2 \cdot 3 \cdot m^3} \text{ approach } \frac{n^3v^3}{1 \cdot 2 \cdot 3 \cdot m^3}.$$

When these limiting values are put in the values of the motions, these values of the motions become identical. Denote the common result by  $b$ , which is either the motion of  $B$  from  $B_1$  for one unit of time with an initial velocity of  $v$ , or the

motion of  $B$  from  $B_1$  for  $v$  units of time with an initial velocity of 1. The value of  $b$  is

$$b = 1 + v + \frac{v^2}{1 \cdot 2} + \frac{v^3}{1 \cdot 2 \cdot 3} + \text{etc.} \quad (62)$$

In this value of  $b$ , make  $v = 1$  and denote the result by  $e$ ;  $e$  is, therefore, the motion of  $B$  from  $B_1$  while  $A$  goes a distance equal to 1, both  $A$  and  $B$  having the same initial velocity. The value of  $e$  is

$$e = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \text{etc.}$$

The numerical value of  $e$ , which is readily computed, is 2.71828+. (See page 10.)

After  $B$  was at  $B_1$  (or 1) its motion for a unit of time with an initial velocity of 1 made its distance from  $B_0$  equal to  $e$ ; hence another unit of time will make its distance  $1 \times e \times e = e^2$ ; and at the end of  $v$  units of time the distance of  $B$  from  $B_0$  will be  $e^v$ ; but this  $= b$ , the distance that  $B$  would be from  $B_0$  at the end of one unit of time, if its velocity when at  $B_1$  was  $v$ . Therefore at the end of the second unit of time, with an initial velocity of  $v$ , the distance of  $B$  from  $B_0$

$$= 1 \times b \times b = b^2 = 1 \times e^v \times e^v = e^{2v};$$

and for  $n$  units of time with an initial velocity of  $v$ , the distance of  $B$  from  $B_0 = b^n = e^{nv}$ ; and hence from the definition,  $b^n = e^{nv}$  is the number whose logarithm is  $n$ . That is, if  $N$  is a number and  $n$  is its logarithm,

$$b^n = N, \text{ where } b = e^v.$$

Taking two numbers,  $N_1$  and  $N_2$ , with logarithms  $n_1$  and  $n_2$ ,

$$N_1 \times N_2 = b^{n_1} \times b^{n_2} = b^{n_1+n_2}.$$

$$N_1 \div N_2 = b^{n_1} \div b^{n_2} = b^{n_1-n_2}.$$

$$N^p = b^{np} \text{ and } \sqrt[p]{N} = b^{\frac{n}{p}}.$$

That is,  $\log (N_1 \times N_2) = \log N_1 + \log N_2,$

$\log (N_1 \div N_2) = \log N_1 - \log N_2,$

$\log N^p = p \log N,$

$\log \sqrt[p]{N} = \log N \div p.$

It will be noticed that

The numbers form a geometric progression ;

Their respective logarithms form an arithmetic progression ;

These systems of logarithms are only particular cases of the more general form of logarithms ;

Logarithms in these systems satisfy the condition :

function  $(ab) = \text{function } (a) + \text{function } (b) ;$

The constant  $(\log 1)$  which was introduced in the Napierian system will be zero in these systems.

**72. Base.** — In the relation  $b^n = N$ , we see that the logarithm  $n$  is that exponent of a number  $b$  corresponding to a number  $N$ . This is expressed by saying that in these systems, a **logarithm is an exponent** ; the number  $b$  which is affected by this exponent is called the **base of the system of logarithms**. It is evident that if  $N$  remains constant and  $b$  is arbitrarily made to change, then  $n$  will also change.

**Relations between logarithms in systems with different bases.** — Let  $x$  be the logarithm of  $a$  in a system whose base is  $b$  (written  $x = \log_b a$ ), and  $z$  be the logarithm of  $b$  in a system whose base is  $a$  (written  $z = \log_a b$ ). Then  $b^x = a$  and  $a^z = b$ .

Now in any of the present systems, taking the logarithms of each side of these equations,

$$x \log b = \log a ; z \log a = \log b ; \text{ whence } xz = 1.$$

$$\text{That is,} \quad \log_b a \times \log_a b = 1. \quad (63)$$

Hence if the logarithm of a number to any base is known, its logarithm to any other base can easily be found.

THE LOGARITHM CORRESPONDING TO A GIVEN  
NUMBER.

**73.** We have heretofore regarded the logarithms as increasing uniformly, and the rate of increase of the numbers continually changing. We will now find a relation giving the increase in the logarithm when the numbers increase uniformly.

Now equation (62) gives the value of  $b = e^v$ , and hence putting  $nv$  for  $v$ ,

$$e^{nv} = b^n = 1 + nv + \frac{n^2 v^2}{1 \cdot 2} + \frac{n^3 v^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

which is the number whose logarithm is  $n$ . Denote this number by  $1 + s$ ; then

$$s = nv + \frac{n^2 v^2}{1 \cdot 2} + \frac{n^3 v^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

Assume now that  $n = As + Bs^2 + Cs^3 + \text{etc.}$ , and substitute this assumed value of  $n$  in the value of  $s$ ; and let us for the present, in squaring and cubing the assumed value of  $n$ , drop powers of  $s$  higher than the third. Equating the coefficients of the like powers of  $s$ , there is obtained

$$Av = 1, \text{ whence } A = \frac{1}{v};$$

$$Bv + \frac{A^2 v^2}{2} = 0, \text{ whence } B = -\frac{1}{2v};$$

$$Cv + ABv^2 + \frac{A^3 v^3}{6} = 0, \text{ whence } C = \frac{1}{3v};$$

and the law of the series being seen, it is unnecessary to repeat the operation retaining higher powers of  $s$ . Hence,

$$n = \log(1 + s) = \frac{1}{v} \left[ s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \frac{s^5}{5} - \text{etc.} \right].$$

Changing the sign of  $s$ ,

$$\log(1 - s) = \frac{1}{v} \left[ -s - \frac{s^2}{2} - \frac{s^3}{3} - \frac{s^4}{4} - \frac{s^5}{5} - \text{etc.} \right].$$

Subtracting these two equations, and remembering that the subtraction of logarithms corresponds to a division of numbers,

$$\log \left( \frac{1+s}{1-s} \right) = \frac{2}{v} \left[ s + \frac{s^3}{3} + \frac{s^5}{5} + \text{etc.} \right].$$

Now let  $s = \frac{p-q}{p+q}$ ; whence  $\frac{1+s}{1-s} = \frac{p}{q}$ .

$$\log \frac{p}{q} = \frac{2}{v} \left[ \frac{p-q}{p+q} + \frac{1}{3} \left( \frac{p-q}{p+q} \right)^3 + \frac{1}{5} \left( \frac{p-q}{p+q} \right)^5 + \text{etc.} \right]. \quad (64)$$

If  $q=1$  and  $p=b$ ,  $\log b = 1$ . Substitute and solve for  $v$ :

$$v = 2 \left[ \frac{b-1}{b+1} + \frac{1}{3} \left( \frac{b-1}{b+1} \right)^3 + \frac{1}{5} \left( \frac{b-1}{b+1} \right)^5 + \text{etc.} \right];$$

whence  $v$  can be computed for any positive base.

Note that if the base is 1,  $v$  is zero; and that if the base is less than 1,  $v$  is negative.

**Modulus.** — The reciprocal of  $v$  is called the modulus of the system of logarithms. Now  $v$  has been shown to be the ratio of the rate of increase in the numbers in a system whose base is  $b$  ( $A$  at  $A_0$ ,  $B$  at  $B_1$ ) to the rate of increase in the numbers in the natural system, the logarithms being conceived to increase uniformly. Hence the reciprocal of  $v$ , or the modulus of the system whose base is  $b$ , is the ratio of rate of increase in the logarithms in the system whose base is  $b$  ( $A$  at  $A_0$ ,  $B$  at  $B_1$ ) to the rate of increase in the logarithms in the natural system, the numbers being conceived to increase uniformly.

It will be seen from (64) that a logarithm in any system is inversely proportional to the value of  $v$  for that system. Now  $v$  is the reciprocal of  $M$ , and hence a logarithm in any system is directly proportional to the modulus of the system. The modulus of the natural system is 1; the logarithms of a number in any other system can therefore be obtained by multiplying the natural logarithm of the number by the modulus of the system.

If in (64)  $M$  is written instead of  $\frac{1}{v}$ , and  $\log q$  is transposed, we obtain

$$\log p = \log q + 2M \left[ \frac{p-q}{p+q} + \frac{1}{3} \left( \frac{p-q}{p+q} \right)^3 + \frac{1}{5} \left( \frac{p-q}{p+q} \right)^5 + \text{etc.} \right].$$

From which the logarithm of any positive number can be found when the logarithm of any other number and also the modulus of the system are known.

**74. The logarithmic curve.**—In the following diagram the abscissæ of points in the curve represent numbers, and the corresponding ordinates represent the natural logarithms of these numbers.

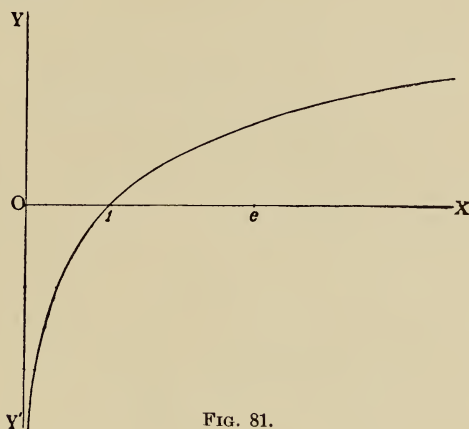


FIG. 81.

**75. Solve the following equations.**—After obtaining the general solution, substitute 2 for  $a$ , 3 for  $b$ , 4 for  $c$ , 5 for  $d$ , 6 for  $e$ , 7 for  $f$ ; obtain the numerical answer in each case.

130.  $a^{bx+c} = d^{ex+f}.$

131.  $a^{bx}c^{dx} = e.$

132.  $ab^x = cd^{x^2}.$

133.  $\begin{cases} a^x = b^y, \\ c^{x+d} = e^{y+f}. \end{cases}$

134.  $\begin{cases} a^{x+y} = b, \\ b^{x-y} = a. \end{cases}$

135.  $a^{x^2}b^xc = 1.$

136.  $a^{b^x} = c.$

## APPENDIX II.

### 1. GONIOMETRY.    2. COMPLEX QUANTITIES. 3. HYPERBOLIC FUNCTIONS.

In the subject of Goniometry we will consider: (1) functions of the sum or difference of angles; (2) the circular measure of an angle; (3) arc functions.

#### Review Chapter I. FUNCTIONS OF ANGLES.

#### 76. Functions of the sum or difference of angles.

Denote the value of angle  $BAC$  by  $x$ , the value of angle  $CAD$  by  $y$ . From any point  $D$  in  $AD$  drop a perpendicular on  $AC$  and on  $AB$ . From  $C$  draw parallels to  $BA$  and  $DE$ , respectively.

The angles  $CDF$  and  $BAC$  are equal because their sides are respectively perpendicular.

$$AD \sin (x + y) = ED = BC + FD.$$

But  $BC = AC \sin x$   
and  $AC = AD \cos y.$

Also  $FD = CD \cos x$  and  $CD = AD \sin y.$

Substituting:  $\sin (x + y) = \sin x \cos y + \cos x \sin y. \quad (65)$

Also,  $AD \cos (x + y) = AE = AB - FC,$

$$AB = AC \cos x = AD \cos x \cos y,$$

$$FC = CD \sin x = AD \sin x \sin y.$$

Hence  $\cos (x + y) = \cos x \cos y - \sin x \sin y. \quad (66)$

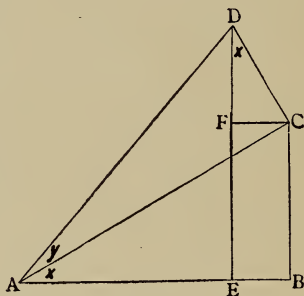


FIG. 82.

Had the figure been drawn with either  $x$  or  $y$  in any other quadrant than the first, the same result would have been obtained.

Had  $y$  been subtracted from  $x$ , the point  $F$  would have been to the right of  $C$ ;  $FC$  would have been drawn from  $F$  to the left, and  $FD$  would have been drawn from  $F$  downward.

$$\text{Hence} \quad \sin(x - y) = \sin x \cos y - \cos x \sin y. \quad (67)$$

$$\text{Also} \quad \cos(x - y) = \cos x \cos y + \sin x \sin y. \quad (68)$$

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Multiply numerator and denominator of this equation by  $\sec x \sec y$ . There is obtained

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$\text{Similarly,} \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

The other functions of  $(x + y)$  and of  $(x - y)$  can be obtained by taking the reciprocals of  $\sin(x \pm y)$ ,  $\cos(x \pm y)$ , and  $\tan(x \pm y)$ .

By adding and subtracting (65) to (68),

$$\sin x \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)], \quad (69)$$

$$\cos x \sin y = \frac{1}{2} [\sin(x + y) - \sin(x - y)], \quad (70)$$

$$\sin x \sin y = -\frac{1}{2} [\cos(x + y) - \cos(x - y)], \quad (71)$$

$$\cos x \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]. \quad (72)$$

In these relations put  $\frac{1}{2}(A + B)$  for  $x$  and  $\frac{1}{2}(A - B)$  for  $y$ .

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B), \quad (73)$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B), \quad (74)$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B), \quad (75)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B), \quad (76)$$

## TRIGONOMETRIC EQUATIONS.

**77.** 1. Solve the equation  $a \sin x + b \cos x = c$ .

Put for  $\cos x$  its value  $\sqrt{1 - \sin^2 x}$ , and solve for  $\sin x$ :

$$\sin x = \frac{ac \pm b \sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}.$$

Or proceed thus: Assume an angle  $y$  such that  $b = a \tan y$ . Substitute in the given equation and multiply by  $\cos y$ :

$$a(\sin x \cos y + \cos x \sin y) = c \cos y.$$

The solution is therefore obtained by obtaining  $y$  from the equation  $\tan y = \frac{b}{a}$  and then finding  $x$  from the equation  $\sin(x + y) = \frac{c}{a} \cos y$ .

The first method of solution has the advantage of expressing  $\sin x$  directly in terms of the given parts; the second method has the advantage of using formulæ adapted to logarithmic computation.

2. Solve the equation  $a \tan x + b \cot x = c$ .

Multiply by  $\tan x$  and arrange:

$$a \tan^2 x - c \tan x + b = 0.$$

From which quadratic equation

$$\tan x = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}.$$

Or proceed thus: from (11) and (9)

$$\tan x = \frac{1 - \cos 2x}{\sin 2x} \text{ and } \cot x = \frac{1 + \cos 2x}{\sin 2x}.$$

Substitute in the given equation and arrange:

$$c \sin 2x + (a - b) \cos 2x = a + b.$$

$$\text{If } \tan y = \frac{a - b}{c}, \quad \sin(2x + y) = \frac{a + b}{c} \cos y.$$

This solution gives a logarithmic solution of the **general equation of the second degree**. Thus in  $ax^2 - cx + b = 0$  divide by  $z$ , and put  $\tan x$  for  $z$ .

The equation  $a \tan x + b \cot x = c$  is obtained.

Hence  $z$  can be found by first finding  $y$ , then  $2x + y$ , then  $x$ ; finally,  $z = \tan x$ .

3. Solve  $a(\sin x + \tan x) = b \sin x (\csc x + \cot x)$ .

This equation can be arranged thus:

$$(\cos x + 1)(a \sin x - b \cos x) = 0,$$

and the roots are obtainable by equating each factor separately to zero.

4. Solve the equation  $a \sin x + b \sin(x + 10^\circ) = c$ .

By applying (65) and arranging, this becomes

$$(a + b \cos 10^\circ) \sin x + b \sin 10^\circ \cos x = c,$$

which is of the form  $a \sin x + b \cos x = c$ .

5. Solve the equation  $\sin x + \sin(x - 10^\circ) = .6$ .

In (73), write  $x$  instead of  $A$  and  $x - 10^\circ$  instead of  $B$ ; there is obtained

$$\sin x + \sin(x - 10^\circ) = 2 \sin(x - 5^\circ) \cos 5^\circ.$$

Comparing with the given equation,

$$\sin(x - 5^\circ) = .3 \sec 5^\circ; \text{ hence } x = 22^\circ 32' \text{ or } 167^\circ 28'.$$

6. Solve the equation  $\sin x \sin(x - 10^\circ) = .6$ .

In (71), put  $x - 10^\circ$  for  $y$  and compare the resulting equation with the given equation

$$\cos(2x - 10^\circ) = \cos 10^\circ - 1.2 = -.2152,$$

$$x = 56^\circ 13' \text{ or } 133^\circ 47'.$$

7. Solve the equation  $\sin x = .6 \sin (x - 10^\circ)$ .

Apply (67),

$$\sin x = .6 \sin x \cos 10^\circ - .6 \cos x \sin 10^\circ.$$

Multiply by  $\csc 10^\circ \csc x$ ; there is obtained

$$.6 \cot x = .6 \cot 10^\circ - \csc 10^\circ.$$

From which  $x = 135^\circ 43'$  or  $315^\circ 43'$ .

8. Solve the equation  $\tan x \tan (x - 10^\circ) = .6$ .

The quotient of (71) and (72) compared with the given equation gives

$$-\frac{\cos (2x - 10^\circ) - \cos 10^\circ}{\cos (2x - 10^\circ) + \cos 10^\circ} = .6$$

which gives  $\cos (2x - 10^\circ) = .25 \cos 10^\circ$ ,

whence  $x = 42^\circ 52'$  or  $147^\circ 8'$ .

**Problems.** — Find the least positive angle satisfying each of the following equations:

137.  $\sin (10^\circ + x) = .6 \sin (20^\circ + x)$ .

138.  $\tan x = .6 \tan (x - 10^\circ)$ .

139.  $\tan (x + 10^\circ) + \tan (x - 10^\circ) = 2 \cot x$ .

140.  $\sin (x + 10^\circ) + \cos (x - 10^\circ) = \cos (x + 10^\circ)$ .

141.  $\cos x \cos (x - 10^\circ) = .6$ .

142.  $32 \cos^2 x + 3 \tan 2x + 3 \cot (45^\circ + x) = 0$ .

143.  $\sin (20^\circ + x) \cos (20^\circ - x) = .6$ .

144.  $\sin 2x + \sin 3x = 3 \sin x$ .

145.  $\sin^2 x - \sin^2 (x - 70^\circ) = .6$ .

146.  $\cot x \tan 2x - \tan x \cot 2x = 2$ .

147.  $\sin (c + x) \csc (c - x) = \cos (a + b) \sec (a - b)$ .

**78. The circular measure of an angle** — Let  $A$  be the center of a circle, and  $CB$  the arc included between the sides of angle  $A = \theta$ .\* With the same center  $A$  and any radius  $AB_1$  draw an arc  $B_1C_1$  also included between the sides of  $A$ .

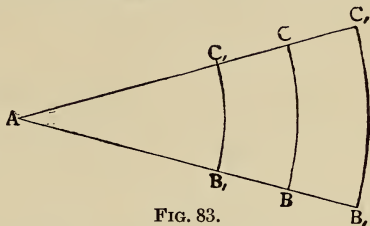


FIG. 83.

The sectors  $ABC$  and  $AB_1C_1$  are similar, and hence

$$\text{arc } BC : \text{radius } AB = \text{arc } B_1C_1 : \text{radius } AB_1.$$

That is, the ratio arc:radius is constant for any angle. This ratio is taken as the measure of angles; its magnitude for any angle is called the circular measure of that angle. That is, in circular measure:

$$\text{angle} = \frac{\text{arc}}{\text{radius}}.$$

The arc of an angle generated by half a revolution ( $180^\circ$ ) is  $\pi$  times the radius. Hence

$$\text{angle of half a revolution} = \pi.$$

The unit angle is obtained by dividing this equation by  $\pi$ .

$$\text{unit angle} = \frac{\text{angle of half a revolution}}{\pi}.$$

This unit angle has been called a **radian**. Its value in degrees can be obtained from this last equation,

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57.296^\circ = 57^\circ 17' 75''.$$

If the angle  $\theta$  in circular measure has  $x^\circ$ ,

$$\theta = \frac{\text{arc of angle } \theta}{\text{radius}} = \frac{\text{degrees in arc of angle } \theta}{\text{degrees in arc equal to radius}}.$$

That is, 
$$\theta = \frac{x^\circ}{57.296^\circ} \text{ and } x = 57.296 \theta.$$

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\* Angles measured in radians will have their values denoted by Greek letters,  $\pi$  (Pi),  $\theta$  (Theta),  $\phi$  (Phi), etc.

A table for converting degrees into radians, or for converting radians into degrees, will be found at the bottom of page vii of the tables.

**Exercises.** — The following table gives the values of various angles, first, in degrees and minutes; second, in radians; third, in terms of an angle of half a revolution:

| ANGLE IN DEGREES. | ANGLE IN RADIANS. | ANGLE IN TERMS OF $\pi$ . | ANGLE IN DEGREES. | ANGLE IN RADIANS. | ANGLE IN TERMS OF $\pi$ . |
|-------------------|-------------------|---------------------------|-------------------|-------------------|---------------------------|
| 172°              | 3.0020            | .9556                     | 144° 9'           | 2.5133            | .8008                     |
| 100°              | 1.7453            | .5556                     | 169° 23'          | 2.9563            | .9401                     |
| 125°              | 2.1566            | .6944                     | 104° 51'          | 1.8300            | .5825                     |
| 179°              | 3.1241            | .9944                     | 204° 12'          | 3.5639            | 1.1344                    |
| 194°              | 3.3859            | 1.0778                    | 195° 33'          | 3.4130            | 1.0864                    |
| 300°              | 5.2360            | 1.6667                    | 274° 10'          | 4.7851            | 1.5231                    |
| 344°              | 6.0039            | 1.9111                    | 289° 50'          | 5.0585            | 1.6102                    |
| 354°              | 6.1784            | 1.9667                    | 344° 19'          | 6.3355            | 2.9667                    |

**79. Arc functions.** — The equation  $N = \sin \theta$  gives the value of  $N$  as a function of  $\theta$ ; it is frequently convenient to express the value of  $\theta$  as a function of  $N$ . The notation employed is:  $\theta = \arcsin N$ .<sup>\*</sup> This will be read,  $\theta$  is the angle (or arc) whose sine is  $N$ . Similarly  $\theta = \arctan N$  is read,  $\theta$  is the angle (or arc) whose tangent is  $N$ ; and similarly for each of the other functions.

It will be seen that if we omit the name of a trigonometric function prefixed to one side of an equation, the name of the corresponding arc function is prefixed to the other side of the equation; and reversely.

It must be remembered, however, that in the equation  $N = \sin \theta$ , there is but one value for  $N$  for each value of  $\theta$ ;

---

<sup>\*</sup> By some authors this is written,  $\theta = \sin^{-1} N$ .

but in the equation  $\theta = \arcsin N$ , there are an infinite number of values of  $\theta$  for any value of the number  $N$ . Thus in the equation  $N = \sin \theta$ , if  $\theta = \frac{1}{6}\pi$  or  $30^\circ$ ,  $N = .5$ ; but in the equation  $\theta = \arcsin N$ , if  $N = .5$ ,  $\theta = [n + (-1)^n \frac{1}{6}]\pi$ ,  $n$  being any whole number.

**Exercises.** — 1. Find  $\theta = \arctan 1$ .

Here  $\tan \theta = 1$ , and  $\theta = (n + \frac{1}{4})\pi$ .

2. Find  $N = \tan(\arccos \frac{1}{2})$ .

Let  $\arccos \frac{1}{2} = \theta$ , then  $\cos \theta = \frac{1}{2}$ ;  $\theta = (2n \pm \frac{1}{3})\pi$ .

Hence  $N = \tan(2n \pm \frac{1}{3})\pi = \pm 1.7321$ .

3. Prove that  $\sin(\arcsin N) = N$ ,  
and that  $\arcsin(\sin \theta) = \theta$ .

4. Prove that  $2 \arctan N = \arctan \frac{2N}{1 - N^2}$ .

5.  $\arcsin N + \arcsin \frac{1}{2}N = \frac{1}{4}\pi$ . Find  $N$ .

6.  $\arcsin 2N - \arcsin N\sqrt{3} = \arcsin N$ . Find  $N$ .

### PROBLEMS IN GONIOMETRY.

**148.** At a distance of 70 feet from the foot of a tower, the angle of elevation of the top is  $21^\circ$  greater than the angle of elevation seen at a distance of 160 feet from the foot of the tower. Find the height of the tower.

**149.** At a distance of 296 feet ( $a$ ) from the base of a tower, the angle of elevation of the top is three times as great as the angle of elevation seen from a point 208 feet ( $b$ ) nearer the base. Find the height of the tower.

**150.** The wind strikes a horseman at an angle of  $50^\circ$  with the road when he is traveling at the rate of 8 miles an hour; and at an angle of  $40^\circ$  when his rate is 12 miles an hour. Find the direction and velocity of the wind.

**151.** From the top of a hill 360 feet above the surrounding plain, an officer sees a column of the enemy's infantry advancing directly toward the hill. The column is known to be one mile long, and at a certain time the whole column is seen by the officer to subtend an angle of  $2^{\circ} 7'$ . At that time what is the distance of the front rank from the middle of the base of the hill?

**152.** A circle has a radius of 20 inches ( $r$ ). From the same point in the circumference two chords are drawn of lengths 10 inches ( $k_1$ ) and 15 inches ( $k_2$ ), respectively. Find the area of the portion of the circle bounded by these chords and by the arc joining the ends of the chords. (Two solutions.)

**153.** A circular field, surrounded by a fence, contains 1 acre. A cow is fastened to the fence by a chain, which, when measured, was found to be 55.53 feet long. Over what area can the cow graze inside the circle?

### COMPLEX QUANTITIES.

**80.** A complex quantity is of the form  $a + bi$ , where  $i$  stands for  $\sqrt{-1}$ ;  $a$  is called the real part of the complex quantity, and  $bi$  is called the imaginary part.

It has already been explained that the multiplication of a segment by  $-1$  reverses the direction of the segment. This reversal may be conceived as being made by means of two half-reversals. Since  $i^2 = -1$ ,  $i$  will be taken as the multiplier which first turns  $OU$  (Fig. 84) into  $OU_1$ , and then turns  $OU_1$  into  $OU_2$ ; multiplying again by  $i$ ,  $OU_2$  is turned into  $OU_3$ ; and another multiplication by  $i$  turns  $OU_3$  into  $OU$ .

Real positive quantities have been conceived as laid horizontally to the right, and real negative quantities to the left. Positive imaginary quantities will be conceived as

being laid vertically upward, and negative imaginaries vertically downward.

Thus the complex quantity  $a + bi$  will be represented by laying off  $OM = a$  to the right of the origin and  $MA_1 = b$  upward from  $M$ . If  $a$  is negative it is laid off to the left of  $O$ , and if  $b$  is negative it is laid off downward.

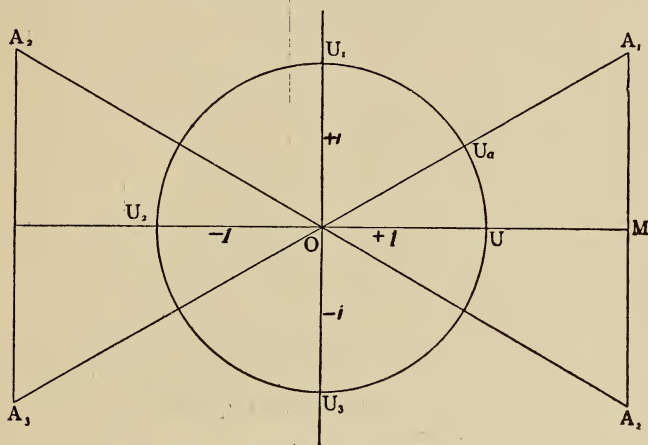


FIG. 84.

Denote the angle  $MOA_1$  by  $\theta$  and the length of the line  $OA_1$  by  $m$ . Then  $a = b \tan \theta = m \cos \theta$  and  $b = m \sin \theta$ .

Hence the complex quantity  $a + bi$  may be written:  $m(\cos \theta + i \sin \theta)$ . The angle  $\theta$  is called the argument of the complex quantity, and  $m$  is called its modulus.

It will be noticed that the segment  $OU (+1)$  is changed into the segment  $OA_1$  by two distinct operations.

*First.* A turning of  $OU$  through the argument  $\theta$  into  $OU_a$ . This turning is therefore equivalent to a multiplication of  $+1$  by  $\cos \theta + i \sin \theta$ .

*Second.* A multiplication of the length of  $OU_a$  by the modulus  $m$ , the positive square root of  $a^2 + b^2$ .

**Addition of complex quantities.** — Suppose it is desired to add the complex quantities  $m_1(\cos \theta + i \sin \theta)$  and  $m_2(\cos \phi + i \sin \phi)$ . Make angle  $MOA = \theta$ ,  $OA = m_1$ ; angle  $NAB = \phi$ ,  $AB = m_2$ ; the required sum is  $OA + AB$ . The sum of the real parts is  $m_1 \cos \theta + m_2 \cos \phi$  ( $OM + AN$ ); and the sum of the imaginary parts is  $i(m_1 \sin \theta + m_2 \sin \phi)$ ; that is,  $MA + NB$ .

Had  $m_2(\cos \phi + i \sin \phi) = OC$  been laid off first and  $m_1(\cos \phi + i \sin \phi) = CB$  been added to it, the result would have been precisely the same. Hence the segment  $OB$  represents the sum both in magnitude and direction.

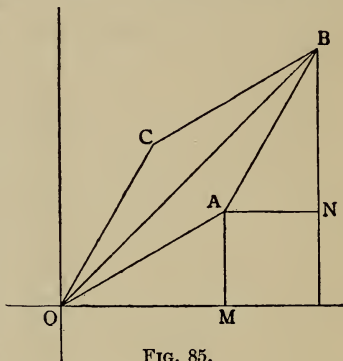


FIG. 85.

A third complex quantity can be added to this sum by drawing from  $B$  a segment, say  $BD$ , representing in magnitude and direction this third complex quantity. Then the segment  $OD$  will represent both in magnitude and direction the sum of the three complex quantities, and so on.

**Subtraction.** — To construct the difference of

$$m_2(\cos \phi + i \sin \phi)$$

and  $m_1(\cos \theta + i \sin \theta)$ ,

lay off the minuend ( $OA$ ); let  $AB$  represent the subtrahend; lay it off backward from  $AB$  to the point

$B_1$ ; then  $OB_1$  will represent the remainder both in magnitude and direction.

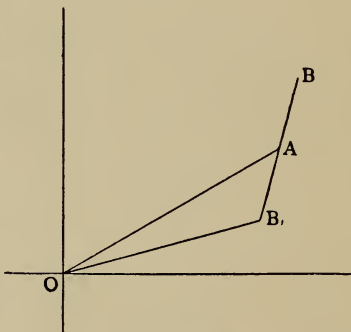


FIG. 86.

**Multiplication.**—The product of the two complex quantities  $m_1(\cos \theta + i \sin \theta)$  and  $m_2(\cos \phi + i \sin \phi)$  is

$$m_1 m_2 [(\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)] \\ = m_1 m_2 [\cos (\theta + \phi) + i \sin (\theta + \phi)], \quad (77)$$

the arguments being added and the moduli multiplied.

Just as  $OU (+1)$  was turned through an angle  $\theta$  and then multiplied by  $m$  when operated upon by the multiplier

$$m(\cos \theta + i \sin \theta),$$

so  $OA$  is turned through an angle  $\phi$  ( $UOB$  or  $AOC$ ) and then multiplied by  $m_2$  when operated upon by the multiplier

$$m_2(\cos \phi + i \sin \phi).$$

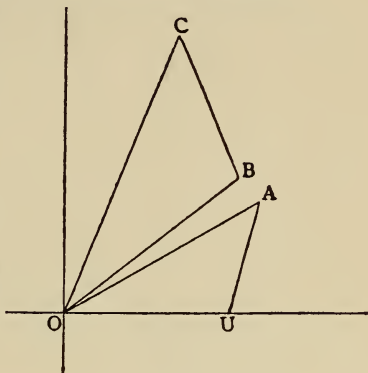


FIG. 87.

If (Fig. 87) angle  $UOA = \theta$ ,  $UOB = \phi$ ,  $UOC = \theta + \phi$ , then  $BOC = UOA$ . Also the length  $OC$  of the modulus  $= m_1 m_2$  is a fourth proportional to 1,  $m_2$ , and  $m_1$ ; that is,  $OU : OB = OA : OC$ . Hence the two triangles  $UOA$  and  $BOC$  are similar.

To construct  $OC$ , therefore, it is sufficient to make angle  $BOC = \theta$  and angle  $B = \text{angle } OUA$ .

If now  $OC$  is operated upon by a new multiplier,

$$m_3(\cos \psi + i \sin \psi),$$

the new argument will be  $\theta + \phi + \psi$ , and the new modulus will be  $m_1 m_2 m_3$ . The geometric construction can be obtained as before.

**Division.**—Rationalizing the denominator of the fraction

$$\frac{m_1(\cos \theta + i \sin \theta)}{m_2(\cos \phi + i \sin \phi)},$$

gives 
$$\begin{aligned} \frac{m_1}{m_2}(\cos \theta + i \sin \theta)(\cos \phi - i \sin \phi) \\ = \frac{m_1}{m_2}[\cos \theta + i \sin \theta][\cos(-\phi) + i \sin(-\phi)] \\ = \frac{m_1}{m_2}[\cos(\theta - \phi) + i \sin(\theta - \phi)], \end{aligned}$$

the arguments being subtracted and the moduli divided.

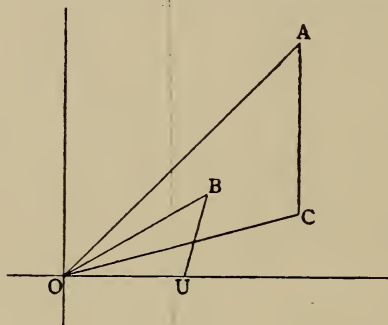


FIG. 88.

To construct the quotient  $OC$ , make angle  $AOC = \phi$  and angle  $A = B$ . Then

the argument  $UOC = UOA - UOB = \theta - \phi$ ;

and since  $OC : OU = OA : OB$ , the length of  $OC =$

the modulus of the quotient  $= \frac{m_1}{m_2}$ .

**Powers and roots.** — The factor  $m(\cos \theta + i \sin \theta)$  repeated  $n$  times gives by (77):

$$[m(\cos \theta + i \sin \theta)]^n = m^n(\cos n\theta + i \sin n\theta). \quad (78)$$

Write  $\frac{\theta}{n}$  instead of  $\theta$ ,  $m$  instead of  $m^n$ , and take the  $n$ th root

$$m^{\frac{1}{n}}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right) = [m(\cos \theta + i \sin \theta)]^{\frac{1}{n}}.$$

To construct the power, lay off, upward from the half-line  $OU$ ,  $n$  angles each equal to  $\angle UOA_1 = \theta$ .

Make  $OA_1 = m$  and join  $UA_1$ . Make angles  $\angle OA_1A_2$ ,  $\angle OA_2A_3$ , etc., each equal to  $\angle OUA_1$ . Then  $OU = 1$ ,  $OA_1 = m$ ,  $OA_2 = m^2$ ,  $OA_3 = m^3$ , etc.

Now the arguments  $\angle UOA_1$ ,  $\angle UOA_2$ ,  $\angle UOA_3$ , etc., increase in arithmetical progression; the segments  $OA_1$ ,  $OA_2$ ,  $OA_3$ , etc., which represent the complex quantities  $m(\cos \theta + i \sin \theta)$ ,  $m^2(\cos \theta + i \sin \theta)^2$ ,  $m^3(\cos \theta + i \sin \theta)^3$ , etc., increase in geometrical progression.

It follows that the arguments  $\theta$ ,  $2\theta$ ,  $3\theta$ , etc., are the logarithms of the corresponding complex quantities. Now denote  $m$  by  $k^\theta$  and take the  $\theta$ th root of  $k^\theta(\cos \theta + i \sin \theta)$ . This root is  $k(\cos 1 + i \sin 1)$ , which is the number whose logarithm is 1, and therefore the base of this system of logarithms.

The curve  $UA_1A_2A_3$  (Fig. 89) is called the **logarithmic spiral**. It will be seen that it may be used as a graphical table of logarithms; the same thing is true of the curve in Fig. 81.

When  $m = 1$ , the points  $A_1$ ,  $A_2$ ,  $A_3$ , etc., will be arranged at equal distances from each other on a circle of unit radius, Fig. 90. In

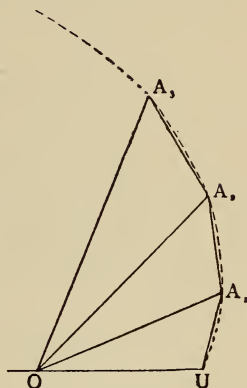


FIG. 89.

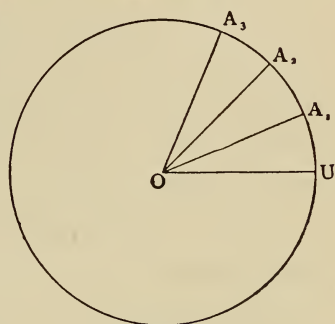


FIG. 90.

this case  $k$  also equals 1 and the base of the system of logarithms is  $\cos 1 + i \sin 1 = .5403 + .8415i$ .

**Construction of roots.**—It has already been shown that the  $n$ th root of  $m(\cos \theta + i \sin \theta)$  was

$$m^{\frac{1}{n}} \left( \cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right),$$

where  $m$  is the positive square root of  $a^2 + b^2$ ;  $m^{\frac{1}{n}}$  will here denote the real positive  $n$ th root of  $m$ . Now the functions of  $2s\pi + \theta$  are the same as those of  $\theta$ , and hence

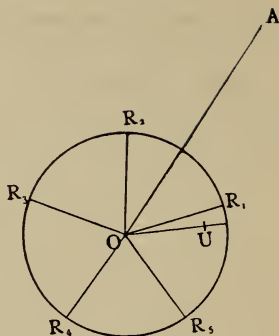


FIG. 91.

$$\left[ m(\cos \theta + i \sin \theta) \right]^{\frac{1}{n}} = m^{\frac{1}{n}} \left( \cos \frac{2s\pi + \theta}{n} + i \sin \frac{2s\pi + \theta}{n} \right),$$

and by taking values of  $s = 0, 1, 2, 3$ , etc., the  $n$  roots are

$$m^{\frac{1}{n}} \left( \cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right),$$

$$m^{\frac{1}{n}} \left( \cos \frac{2\pi + \theta}{n} + i \sin \frac{2\pi + \theta}{n} \right),$$

$$m^{\frac{1}{n}} \left( \cos \frac{4\pi + \theta}{n} + i \sin \frac{4\pi + \theta}{n} \right), \text{ etc.}$$

Taking  $\theta = \frac{5}{18}\pi = 50^\circ$ ,  $m = 3.052$ ,  $n = 5$ , the roots are

$$1.25(\cos 10^\circ + i \sin 10^\circ), \quad 1.25(\cos 82^\circ + i \sin 82^\circ),$$

$$1.25(\cos 154^\circ + i \sin 154^\circ), \quad 1.25(\cos 226^\circ + i \sin 226^\circ),$$

$$1.25(\cos 298^\circ + i \sin 298^\circ),$$

giving the points  $R_1, R_2, R_3, R_4, R_5$ .

By taking  $m=1$  and  $x=0$ , the  $n$  roots of  $+1$  are obtained.

By taking  $m=1$  and  $x=\pi$ , the  $n$  roots of  $-1$  are obtained.

By taking  $m=1$  and  $x=\frac{1}{2}\pi$ , the  $n$  roots of  $i$  are obtained.

By taking  $m=1$  and  $x=\frac{3}{2}\pi$ , the  $n$  roots of  $-i$  are obtained.

**81. Values of functions of multiple angles.**—Put 1 for  $m$  in (78):  $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$ . (79)

Expand by the binomial formula; equate the imaginary parts and also the real parts:

$$\sin n\theta = n \cos^{n-1}\theta \sin \theta - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cos^{n-3}\theta \sin^3 \theta + \text{etc.}$$

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2}\theta \sin^2 \theta + \text{etc.}$$

Thus to obtain the functions of  $3\theta$ , make  $n = 3$ ,

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta, \quad (80)$$

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = 4 \cos^3 \theta - 3 \cos \theta.$$

**82. Exponential value of  $\cos \theta + i \sin \theta$ .**—When  $\theta$  is zero,  $\cos \theta + i \sin \theta$  is 1. Now let us imagine a point generating a circle; when the angle is zero, counted from a horizontal initial line, the horizontal component of the motion of the generating point is zero, and the vertical component of its motion is equal to the motion in the arc. Therefore when  $\theta$  is at zero, the rate of increase of  $\cos \theta + i \sin \theta$  is  $i$  times the rate of the generating point. If we assume the generating point to move a unit distance in unit time, the rate of increase ( $\theta = \text{zero}$ ) of  $\cos \theta + i \sin \theta$  is  $i$ .

Now  $\theta$  is the logarithm of  $\cos \theta + i \sin \theta$ ; therefore, by the theory of logarithms,  $\cos \theta + i \sin \theta = e^{\theta v}$ ,  $v$  being the rate of increase in the number  $\cos \theta + i \sin \theta$ , when  $\theta$  is zero and increasing at unit rate.

$$\text{Therefore} \quad \cos \theta + i \sin \theta = e^{\theta i};$$

$$\text{Changing the sign of } \theta: \cos \theta - i \sin \theta = e^{-\theta i};$$

$$\text{whence } \sin \theta = \frac{1}{2i} (e^{\theta i} - e^{-\theta i})$$

$$= \frac{\theta}{1} - \frac{\theta^3}{1 \cdot 2 \cdot 3} + \frac{\theta^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{\theta^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.} \quad (81)$$

$$\cos \theta = \frac{1}{2} (e^{\theta i} + e^{-\theta i})$$

$$= 1 - \frac{\theta^2}{1 \cdot 2} + \frac{\theta^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\theta^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.} \quad (82)$$

## HYPERBOLIC FUNCTIONS.

**83. Definition of an equilateral hyperbola.** — Suppose a double cone (called a cone of two nappes) with vertex at  $V$ . If this cone is cut through the axis, two angles,  $BVE$  and  $DVC$ , will be formed. If the cone is such that this angle is  $90^\circ$ , and the cone is again cut by a plane  $OA$  normal to the plane  $BVE$ , this new section is bounded by a curve of two branches  $QAR$  and  $Q_1A_1R_1$ ; this curve is called an equilateral hyperbola. The points  $A$  and  $A_1$  are called the vertices of the hyperbola, the point  $O$ , midway between them, is called the center of the hyperbola, and  $AA_1$  is called the axis of the hyperbola.

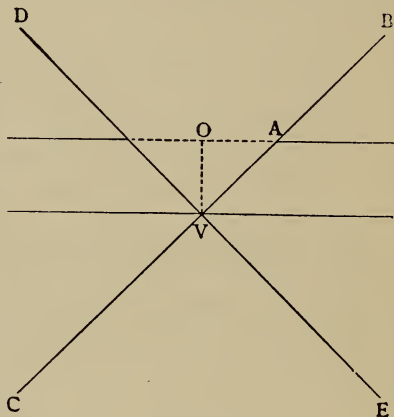


FIG. 92.

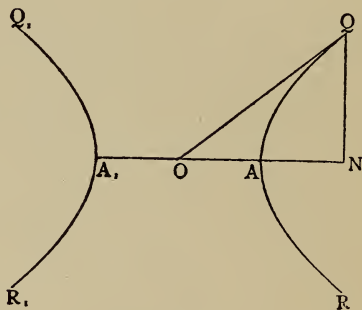


FIG. 93.

The two following properties of the equilateral hyperbola are proved in treatises dealing with this curve.

*First.* If  $Q$  is a point on the curve, and  $ON$  is the abscissa of  $Q$ ,

$$ON^2 - NQ^2 = OA^2.$$

*Second.* Area segment  $AQN$  of the hyperbola

$$= \frac{1}{2} ON \times NQ - \frac{1}{2} OA^2 \log_e \left( \frac{ON + NQ}{OA} \right).^*$$

Now the sector  $OAQ$  = triangle  $ONQ$  - segment  $AQN$ .

$$\text{Hence sector } OAQ = \frac{1}{2} OA^2 \log_e \left( \frac{ON + NQ}{OA} \right).$$

The quantity  $\log_e \left( \frac{ON + NQ}{OA} \right)$  will be denoted by  $u$ ; and the semi-axis  $OA$  will be denoted by  $a$ .

$$\text{Hence Sector } OAQ = \frac{1}{2} a^2 u.$$

**84. Definitions of the hyperbolic functions.** — Let  $O$  be the center of an equilateral hyperbola,  $OA (= a)$  its semi-axis,  $Q$  any point on the curve,  $ON$  the abscissa of  $Q$ , and  $NQ$  the ordinate. With a center  $O$  and a radius  $a$ , describe a circumference; let  $P$  be a point on the circumference; let the length of arc  $AP$  be equal to  $a\theta$ ; then  $\theta$  represents the circular measure of the angle  $AOP$ . The area of the circular sector  $AOP$  equals its arc times  $\frac{1}{2}$  the radius =  $\frac{1}{2} a^2 \theta$ .

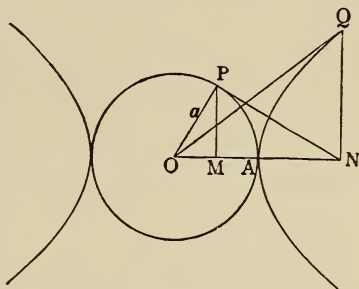


FIG. 94.

\* 1. The equation of the two generators of the cone are  $x - y = 0$  and  $x + y = 0$ , their product being  $x^2 - y^2 = 0$ . As these generators revolve around the axis of  $X$ ,  $x$  remains constant and  $y^2$  becomes  $y^2 + z^2$ . Hence the equation of the double cone is  $x^2 - y^2 - z^2 = 0$ . If now the cone is cut by the plane  $z = a$ , the projection of the section on the  $xy$ -plane is  $x^2 - y^2 = a^2$ .

2. The integral of  $y dx$  or  $\sqrt{x^2 - a^2} dx$  is

$$\frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log_e (x + \sqrt{x^2 - a^2}).$$

Taking  $x$  for the upper limit and  $a$  for the lower, and putting  $y$  for  $\sqrt{x^2 - a^2}$ , the area is  $\frac{1}{2} xy - \frac{1}{2} a^2 \log_e \left( \frac{x + y}{a} \right)$ .

From  $P$ , drop a perpendicular  $PM$  on  $OA$ . Then  $PM = a \sin \theta$ ,  $OM = a \cos \theta$ , and  $\frac{PM}{OM} = \tan \theta$ , and so on. From these properties of  $PM$  and  $OM$ , the trigonometric functions have been also called circular functions.

From analogy, certain ratios of the semi-axis, the abscissa, and ordinate of a point on an equilateral hyperbola are called hyperbolic functions.

They are defined as follows: Let  $u$  be such a quantity that

$$\text{Area sector } OAQ = \frac{1}{2} a^2 u. \quad \text{Then}$$

1. The ratio of  $NQ$  to  $OA$  is called the **hyperbolic sine** of  $u$ . Hyperbolic sine  $u$  will be written  $\sinh u$  and will be read "shin  $u$ ." It follows that  $QN = a \sinh u$ .

2. The ratio of  $ON$  to  $OA$  is called the **hyperbolic cosine** of  $u$ ; written and pronounced  $\cosh u$ . It follows that  $ON = a \cosh u$ .

3. The ratio of  $\sinh u$  to  $\cosh u$  ( $NQ : ON$ ) is the **hyperbolic tangent** of  $u$  ( $\tanh u$ ).

4. The reciprocal of  $\cosh u$  ( $OA : ON$ ) is the **hyperbolic secant** of  $u$  ( $\text{sech } u$ ).

5. The reciprocal of  $\sinh u$  ( $NQ : OA$ ) is the **hyperbolic cosecant** of  $u$  ( $\text{csch } u$ ).

6. The reciprocal of  $\tanh u$  ( $ON : NQ$ ) is the **hyperbolic cotangent** of  $u$  ( $\text{coth } u$ ).

**85. Exponential values of hyperbolic functions.**—By the property of the equilateral hyperbola,

$$ON^2 - NQ^2 = OA^2.$$

Dividing by  $OA^2$ ,  $\cosh^2 u - \sinh^2 u = 1$ .

$$\begin{aligned} \text{But since } u &= \log_e \left( \frac{ON + NQ}{OA} \right), \\ \cosh u + \sinh u &= e^u. \end{aligned}$$

Hence, by division,

$$\cosh u - \sinh u = e^{-u}.$$

Therefore, by subtraction and addition,

$$\sinh u = \frac{1}{2} (e^u - e^{-u}), \quad (83)$$

$$\cosh u = \frac{1}{2} (e^u + e^{-u}). \quad (84)$$

#### GENERAL RELATIONS.

##### 86. Relations of hyperbolic and circular functions.

Now 
$$\frac{1}{2} (e^u - e^{-u}) = -i \left[ \frac{1}{2i} (e^{ui} - e^{-ui}) \right].$$

Hence by (81) and (83):  $\sinh u = -i \sin ui. \quad (85)$

Also, 
$$\frac{1}{2} (e^u + e^{-u}) = \frac{1}{2} (e^{ui} + e^{-ui}).$$

Hence by (82) and (84):  $\cosh u = \cos ui. \quad (86)$

##### Periodicity of hyperbolic functions.

Now 
$$\begin{aligned} -i \sin ui &= -i \sin (ui - 2n\pi) \\ &= -i \sin (u + 2n\pi i) i. \end{aligned}$$

By (85),  $\sinh u = \sinh (u + 2n\pi i). \quad (87)$

Similarly,  $\cosh u = \cosh (u + 2n\pi i). \quad (88)$

**The addition formulæ.** — From (85), (86), (65), and (66) :

$$\begin{aligned} \sinh (u + v) &= -i \sin (ui + vi) \\ &= -i (\sin ui \cos vi + \cos ui \sin vi) \\ &= \sinh u \cosh v + \cosh u \sinh v. \end{aligned} \quad (89)$$

$$\begin{aligned} \cosh (u + v) &= \cos (ui + vi) = \cos ui \cos vi - \sin ui \sin vi \\ &= \cosh u \cosh v + \sinh u \sinh v. \end{aligned} \quad (90)$$

**Functions of multiples of  $u$ .** — From (89) and (90) there is obtained

$$\begin{aligned} \sinh 2u &= 2 \sinh u \cosh u, \\ \cosh 2u &= \cosh^2 u + \sinh^2 u, \\ \sinh 3u &= 4 \sinh^3 u + 3 \sinh u, \end{aligned} \quad (91)$$

$$\cosh 3u = 4 \cosh^3 u - 3 \cosh u. \quad (92)$$

**87. Longitude of  $u$ .** — With center  $N$  and radius equal to  $NQ$ , draw an arc intersecting the circle in  $P$ . Join  $NP$  and  $OP$ . Since

$$ON^2 - NQ^2 = a^2,$$

$$ON^2 - NP^2 = OP^2.$$

Therefore, the angle  $OPN$  is a right angle. Denote the angle  $PON$  by  $x$ .

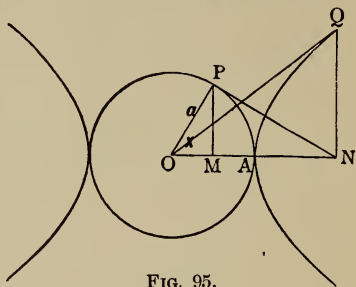


FIG. 95.

$$\sinh u = NQ : OA = NP : OP = \tan x, \quad (93)$$

$$\cosh u = ON : OA = ON : OP = \sec x. \quad (94)$$

In (11) put  $90^\circ + x$  for  $x$ ; then

$$\tan x + \sec x = \tan(45^\circ + \tfrac{1}{2}x).$$

$$\text{Therefore } \tan(45^\circ + \tfrac{1}{2}x) = \sinh u + \cosh u = e^u.$$

$$\text{Hence } \log \tan(45^\circ + \tfrac{1}{2}x) = u \log e = .43429 u.$$

From which for any value of  $u$ ,  $x$  can be found, and thence in (93) and (94),  $\sinh u$  and  $\cosh u$  can be found.

The angle  $x$  was called by Gudermann the longitude of  $u$ . Some later writers, however, have referred to it as the Gudermannian function of  $u$ .

A table at the bottom of page vii gives values of  $u$  corresponding to values of  $x$ , the longitude of  $u$ , within limits which admit of interpolation.

**88. Solution of cubic equations.** — Consider the cubic equation  $w^3 \pm aw = b$ .

**First.** The coefficient of  $w$  is positive.

$$\text{Let } w = 2 \sinh u \sqrt{\frac{a}{3}}. \quad (95)$$

The given equation reduces to

$$\sinh^3 u + \frac{3}{4} \sinh u = \frac{3b}{8a} \sqrt{\frac{3}{a}}.$$

$$\text{Hence by (91), } \sinh 3u = \frac{3b}{2a} \sqrt{\frac{3}{a}}. \quad (96)$$

Denote three values obtained from this equation by  $3u$ ,  $3u + 2\pi i$ ,  $3u - 2\pi i$ .

By (89),

$$\begin{aligned}\sinh(u \pm \tfrac{2}{3}\pi i) &= \sinh u \cosh \tfrac{2}{3}\pi i \pm \cosh u \sinh \tfrac{2}{3}\pi i \\ &= \sinh u \cos 120^\circ \pm i \cosh u \sin 120^\circ.\end{aligned}$$

Hence, to solve the equation under consideration, find  $u$  from (96); then by (95) the three roots of  $w$  are

$$2 \sinh u \sqrt{\frac{a}{3}}, \text{ and } \sqrt{\frac{a}{3}} (-\sinh u \pm i\sqrt{3} \cosh u).$$

**Second.** Consider the equation  $w^3 - aw = b$ .

**CASE I.**  $27b^2$  is greater than  $4a^3$ .

Let 
$$w = 2 \cosh u \sqrt{\frac{a}{3}}.$$

The given equation reduces to

$$\cosh^3 u - \tfrac{3}{4} \cosh u = \frac{3b}{8a} \sqrt{\frac{3}{a}}.$$

Hence by (92), 
$$\cosh 3u = \frac{3b}{2a} \sqrt{\frac{3}{a}}. \quad (97)$$

Find  $u$  from (97); then the three roots are

$$2 \cosh u \sqrt{\frac{a}{3}}, \text{ and } \sqrt{\frac{a}{3}} (-\cosh u \pm i\sqrt{3} \sinh u).$$

**CASE II.**  $27b^2$  is less than  $4a^3$ . — This is the irreducible case of Tartaglia's solution, in which all the roots are real.

Let 
$$w = 2 \sin x \sqrt{\frac{a}{3}}.$$

The given equation reduces to

$$\sin^3 x - \tfrac{3}{4} \sin x = \frac{3b}{8a} \sqrt{\frac{3}{a}}.$$

Hence by (80), 
$$\sin 3x = -\frac{3b}{2a} \sqrt{\frac{3}{a}}. \quad (98)$$

Find  $x$  from (98); then the three roots of  $w$  are

$$2 \sin x \sqrt{\frac{a}{3}}, \quad 2 \sin(x \pm 120^\circ) \sqrt{\frac{a}{3}}.$$

**Exercises.** — 1. Find the value of  $i^i$ .

Raise to the  $i$  power the equation  $\cos \theta + i \sin \theta = e^{i\theta}$ .

$$(\cos \theta + i \sin \theta)^i = e^{-\theta}.$$

Make  $\theta = (2n + \frac{1}{2})\pi$ .

$$i^i = e^{-\frac{1}{2}\pi} \times e^{2n\pi} = .2079 e^{2n\pi},$$

where  $n$  is any whole number; when  $n$  is zero,  $i^i = .2079$ .

2. Solve by Horner's Method the equation

$$x^3 - \frac{3}{4}x + \frac{1}{4}\sin 30^\circ = 0,$$

and thus find  $\sin 10^\circ$ ,  $\sin 50^\circ$ , and  $\sin 130^\circ$ .

3. By substituting  $\frac{\pi}{18}$  for  $\theta$  in (81), find  $\sin 10^\circ$ .

4. Solve the following cubic equations:

$$x^3 + 24x = 511$$

$$x^3 - 6x = 9$$

$$x^3 - 9x = 10$$

$$x^3 + 54x = 288$$

$$x^3 - 7x = 90$$

$$x^3 - 7x = 6$$

$$x^3 + 72x = 400$$

$$x^3 - 48x = 520$$

$$x^3 - 39x = 70$$

## PROBLEMS FOR EXAMINATION.

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[No answers are given for these problems.]

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### RIGHT TRIANGLES.

**154.** A person standing at a distance of 143 feet from the middle of the base of a tower finds the altitude of the tower to be  $43^{\circ} 10'$ . Find the height of the tower, the eye of the observer being 5 feet from the ground.

**155.** From the top of a lighthouse 127 feet above the level of the sea, the angle of depression of a boat was seen to be  $10^{\circ} 20'$ . Find the distance of the boat from the foot of the lighthouse.

**156.** At what angle do we ascend a regular acclivity 6.5 miles long, attaining an altitude at the summit of 4268 feet?

**157.** The horizontal distance between two columns is 124 feet, and the straight line joining the tops of the columns makes an angle of  $12^{\circ} 40'$  with the horizon. Find the distance between the tops.

**158.** A canal boat is drawn along the middle of a straight canal by two men hauling upon ropes 60 feet and 70 feet long, respectively. Find the ratio of the strains upon them, the canal being 20 feet wide.

**159.** A row-boat, pulled at the rate of 5 miles an hour, is required to cross a river 1.3 miles wide to a point which is .78 mile lower down the stream than the point of starting. What angle must the direction of the boat's head make with the bank, supposing the stream to run at the rate of 4 miles an hour?

**160.** If  $d$  is the distance (in degrees) of the sun from the equator, show that the altitude of the sun at midday is  $90^\circ + d - l$ , where  $l$  is the latitude of the place. Hence, find the length of the shadow at midday of a tower 120 feet high, the latitude of the tower being  $40^\circ$ , and the sun being vertically above a point in  $10^\circ$  north latitude.

**161.** At midday of the 21st of March, the length of the shadow of a 12-foot pole was 10.6 feet. What was the latitude of the place?

**162.** *The problem of Eratosthenes.* At midday, when the sun was vertically over Syene, the shadow of a vertical rod at Alexandria was measured, the shadow being found to be .13 times the length of the rod. Taking Alexandria to be 5000 stadia due north of Syene, find the circumference of the earth.

**163.** The latitude of Berlin is  $52^\circ 30'$ . Taking the earth to be a sphere whose radius = 3960 miles, find the radius and length of  $1^\circ$  of the parallel passing through Berlin.

**164.** What will be the visual angle of a sphere 6 feet in diameter, if the eye of the observer is placed 7 feet from its center?

**165.** On a hill over the sea is a 60 feet high tower. From a boat, the angle of elevation of the foot of the tower is  $14^\circ$ , and of the top of the tower is  $20^\circ$ . Find the height of the hill.

**166.** A flagpole, 20 feet high, is on the top of a tower. At a horizontal distance of 420 feet from the foot of the tower, the angle of elevation of the highest point of the flagpole is seen to be  $15^\circ 20'$ . Find the height of the tower.

**167.** *The lune of Hippocrates.* A semi-circumference and a quadrant concave in the same direction are drawn to meet at their extremities so as to inclose space; the radius of the semi-circumference is 10 feet; find the area inclosed.

**168.** On one side of an angle  $A = 50^\circ$ , a distance  $AB = 10$  feet is taken. From  $B$ , a perpendicular  $BC$  is let fall upon the other side of the angle; from  $C$ , a perpendicular  $CD$  is let fall upon the first side; from  $D$  a perpendicular  $DE$  is let fall upon the second side; and so on, *ad infinitum*. Find the sum of all the perpendiculars.

Put  $A$  instead of  $50^\circ$  and  $a$  instead of 10 feet; then obtain the general solution of this problem.\*

**169.** A circle has a radius of 22 inches. From the same point in the circumference, two chords are drawn of lengths 10 inches and 17 inches respectively. What angle is formed by these chords?

**170.** Two towers stand 100 feet apart on the same horizontal plane. An observer in line with them, and 150 feet from the nearer one, sees the angle of elevation of both of them to be the same. He then walks directly toward them a distance of 130 feet, when he finds the angle of elevation of the nearer one to be twice as great as that of the farther one. Find the heights of the towers.

Before performing the computation for this problem, obtain the general solution of it.

**171.** The top of a hill is 90 feet higher than the foot of a tower; from the top of the hill the angle of depression of the top of the tower is  $5^\circ 43\frac{1}{2}'$ . The horizontal distance from the tower to the hill is 399 feet. Find the height of the tower.

**172.** I bought a lot having the shape of a right triangle; the hypotenuse = 875 feet, and one of the acute angles =  $37^\circ 53'$ . How much was paid for the land if the price was 2.7 cents per square foot?

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\* If for the data of a problem, literal values are assumed instead of numerical values, the answer to the problem expressed in terms of these literal values is called the **general solution** of the problem.

## OBLIQUE TRIANGLES.

**173.** To measure the distance between two points  $A$  and  $B$  separated by a pond, a third point  $C$  was chosen.  $AC$  was found to be 545 feet,  $CB = 305$  feet, and the angle  $ACB = 67^\circ$ . Find  $AB$ .

**174.** From a point on the bank of a river, an observer goes directly away from the river 165 yards up a slope inclined at an angle of  $12^\circ 12'$  to the horizontal; at this point the angle of depression of an object on the opposite bank and in the same plane with the first position was found to be  $2^\circ 10'$ . Find the breadth of the river.

**175.** *The range finder.* From a fort a torpedo-boat is observed to be exactly east; eighty seconds afterward it is exactly northeast, and eighty seconds after that it is N.  $22^\circ 30'$  E. The torpedo-boat is going at a uniform rate in a straight line. Find its bearing at the end of the next eighty seconds.

**176.** A vertical tower stands on a plane inclined  $11^\circ 20'$  to the horizontal. Measuring from the base of the tower down the slope an oblique distance of 59.18 feet, the tower was found to subtend an angle of  $38^\circ$ . Find the height of the tower.

**177.** Two inaccessible points  $X$  and  $Y$  are observed from two stations  $A$  and  $B$ , 500 feet apart ( $A$  and  $X$  on the left). The angle  $XAY = 42^\circ$ , angle  $YAB = 67^\circ$ , angle  $ABX = 39^\circ$ , angle  $XB Y = 47^\circ$ . Find  $XY$ .

**178.** A house is seen from two ships  $A$  and  $B$  which are 300 feet apart. At  $A$ , the angle of elevation of the house is found to be  $28^\circ 20'$  and the horizontal angle from it to the ship  $B = 48^\circ 40'$ . At  $B$ , the horizontal angle from the house to the ship  $A$  is found to be  $20^\circ 15'$ . Find the height of the house above the level of the water.

**179.** To find the length  $BC$  of a lake running due north and south ( $C$  north of  $B$ ), a point  $D$  was taken 530 feet north of  $C$  and a point  $A$  was found due west of  $B$ . It was found that  $AC = 1970$  feet and  $AD = 2400$  feet. Find the length of the lake.

**180.** To find the distance between two inaccessible points  $A$  and  $B$ , an observer chose a point  $C$  in the same plane and found angle  $ACB = 113^\circ$ . Then he walked on the continuation of  $AC$  to a point 125 yards from  $C$ , where the angle subtended by  $AB$  was found to be one-half of angle  $ACB$ . Then he walked on the continuation of  $BC$  173 yards from  $C$  and again found the angle subtended by  $AB$  to be one-half of angle  $ACB$ . Find  $AB$ .

**181.** Given the three sides of a triangle to find the length from the vertex to a point of trisection of the base. Take the base ( $c$ ) = 200 feet,  $a = 205$ ,  $b = 85$ , and find the length to the point of trisection near  $a$ .

**182.** How many square miles of the earth's surface can an observer see if stationed at the top of Mt. Everest, whose altitude is 29,000 feet? The radius of the earth = 3960 miles.

**183.** A lighthouse 60 feet high is on a cliff over the sea. From the top of the lighthouse the angle of depression of a boat was seen to be  $16^\circ 20'$ ; and from the bottom of the lighthouse, the angle of depression of the boat was  $12^\circ 10'$ . Find the height of the cliff and the distance of the ship from the foot of the cliff.

**184.** Wishing to know the breadth of a river from a point  $X$  to a point  $A$ , an observer measures in the prolongation of  $XA$  a distance  $AB = 500$  feet; then from  $B$ , he measures at right angles to  $XAB$  a distance  $BC = 700$  feet; from the point  $C$ , the angle  $XCA$  was found to be  $24^\circ 20'$ . Find the breadth of the river.

**185.**  $P$ ,  $Q$ , and  $R$  are three known points in a straight line;  $PQ = 104$ ,  $QR = 296$ ;  $PQ$  and  $QR$  are observed to subtend equal angles  $= 46^\circ 50'$  at a certain point  $S$ . Find  $SQ$ .

**186.** From the bottom of a wall 20 feet high, the angle of elevation of the top of a house  $= 40^\circ 20'$ ; at the top of the wall the angle of elevation of the same is  $19^\circ 40'$ . Find the height of the house.

**187.** A person at a point  $A$ , wishing to know the distance of an inaccessible object  $C$  on the opposite side of a river, lays off a base  $AB = 898.8$ ; he then measures the angle  $CAB = 63^\circ 20'$  and angle  $CBA = 52^\circ 40'$ . Find  $AC$ .

**188.** Of three towns the second is 11.6 miles from the first, the third is 10.5 miles from the second, and the first is 14.3 miles from the third. Find the angles of the triangle whose vertices are the three towns.

**189.** From the deck of a ship, 10 feet above the sea, the angle of elevation of the top of a cliff is  $20^\circ 20'$ ; from the top of the mast 65 feet higher, the angle is  $12^\circ 30'$ . Find the height of the cliff from the sea and the distance of the ship from the foot of the cliff.

**190.** From the top of a tower 60 feet high, a column subtends an angle of  $28^\circ$ ; from the base of the tower, the column subtends an angle of  $40^\circ$ . Find the height of the column.

**191.** A balloon is ascending vertically with a uniform velocity. When it is one mile high, its angle of elevation is  $32^\circ$ ; twenty minutes later, the elevation is  $54^\circ$ . How fast is the balloon rising?

**192.** At two points, 70 and 50 feet from the foot of a tower, a flagstaff on the top of the tower subtends the same angle, viz.,  $3^\circ$ . Find the length of the flagstaff.

## FUNCTIONS OF ANGLES.

Find a function of an angle of the first quadrant equivalent to each of the following functions:

- |                         |                             |                             |
|-------------------------|-----------------------------|-----------------------------|
| 193. $\sin 283^\circ$ . | 201. $\sin 324^\circ$ .     | 209. $\sec 247^\circ 42'$ . |
| 194. $\cot 196^\circ$ . | 202. $\csc 562^\circ$ .     | 210. $\csc 176^\circ 28'$ . |
| 195. $\cos 344^\circ$ . | 203. $\cos 897^\circ$ .     | 211. $\cot 312^\circ 53'$ . |
| 196. $\sec 127^\circ$ . | 204. $\tan 631^\circ$ .     | 212. $\sec 228^\circ 12'$ . |
| 197. $\tan 274^\circ$ . | 205. $\tan 189^\circ 49'$ . | 213. $\tan 142^\circ 29'$ . |
| 198. $\cos 245^\circ$ . | 206. $\cos 128^\circ 11'$ . | 214. $\cos 218^\circ 47'$ . |
| 199. $\csc 144^\circ$ . | 207. $\sin 212^\circ 14'$ . | 215. $\sin 374^\circ 55'$ . |
| 200. $\cot 133^\circ$ . | 208. $\cot 321^\circ 37'$ . | 216. $\csc 524^\circ 19'$ . |

Reduce each of the following expressions to its simplest form:

217.  $a \cos (180^\circ - x) + b \cos (90^\circ + x) \cot x$ .
218.  $k \sin x \cot x \tan (180^\circ + x) \sec (90^\circ - x)$ .
219.  $\tan (180^\circ - x) \cot (270^\circ - x) + \cot (90^\circ + x) \cot x$ .
220.  $\cos (90^\circ + x) \sin (360^\circ - x) - \sin (270^\circ - x) \cos (-x)$ .
221.  $(a + b)^2 \sec (360^\circ - x) - (a - b)^2 \csc (90^\circ + x)$ .
222.  $\sec (90^\circ + x) \csc (360^\circ - x) + \tan (90^\circ + x) \cot x$ .
223.  $\tan x \cot x + \tan (-y) + \tan (180^\circ + y)$ .
224.  $\sin (180^\circ - x) \cos (90^\circ + x) \csc (180^\circ + x) \csc x$ .
225.  $(a^2 + b^2) \sec (270^\circ - x) + (a^2 - b^2) \csc (180^\circ + x)$ .
226.  $a \sin (360^\circ - x) \tan (270^\circ + x) \sec (180^\circ - x)$ .
227.  $2 \cot 2x \cos x + 2 \sin x - \csc x$ .
228.  $a^2 \sin 90^\circ + 2ab \cos 180^\circ + b^2 \sec 0^\circ$ .
229.  $m \sin 90^\circ - n \cos 360^\circ + (m - n) \cos 180^\circ$ .

Prove the truth of the following equations :

$$230. \frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x.$$

The proof consists in reducing the equation to an identity. Thus, by clearing of fractions, both sides of the equation are  $\sin x + \tan x$ . By writing the work in reverse order, a direct proof of the given equation is obtained.

$$231. \tan x = \frac{(1 + \sin x)(1 + \sec x)}{(1 + \cos x)(1 + \csc x)}.$$

$$232. \csc x \sec x = \frac{\sec x \cot x - \csc x \tan x}{\cos x - \sin x}.$$

$$233. \tan x - \cot x = \frac{1 - 2 \cos^2 x}{\sin x \cos x}.$$

$$234. \tan x + \cot x = \cot x \sec^2 x.$$

$$235. \sin^2 x + \tan^2 x = \sec^2 x - \cos^2 x.$$

$$236. \sec x - \cos x = \sin x \tan x.$$

$$237. \frac{\sin x + \cos x}{\sin x - \cos x} = \frac{\sec x + \csc x}{\sec x - \csc x}.$$

$$238. (\sin x + \tan x)(\cos x + \cot x) = (1 + \sin x)(1 + \cos x).$$

$$239. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \cot x}{1 + \cot^2 x}.$$

$$240. \tan 2x = \frac{1}{1 - \tan x} - \frac{1}{1 + \tan x}.$$

$$241. \sec 2x = \frac{\sec^2 x}{1 - \tan^2 x} = \frac{\sec x \csc x}{\cot x - \tan x}.$$

$$242. \csc 2x = \frac{1}{2} \cot x \cos^2 x = \frac{1}{2} (\tan x + \cot x).$$

$$243. \cot 2x = \frac{1}{2} (\cot x - \tan x) = \frac{1}{2} \sec x \csc x - \tan x.$$

$$244. \cos 2x = \frac{\cot^2 x - 1}{\cot^2 x + 1} = \frac{\cot x - \tan x}{\cot x + \tan x}.$$

$$245. \sin(x+y) = (\tan x + \tan y) \cos x \cos y.$$

$$246. \tan(x+y) = \frac{\sin^2 x - \sin^2 y}{\sin x \cos x - \sin y \cos y}.$$

$$247. \sec(x+y) = \frac{\sec x \sec y}{1 - \tan x \tan y}.$$

$$248. \frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}.$$

$$249. \tan x \tan y = \frac{\tan x + \tan y}{\cot x + \cot y}.$$

$$250. \tan(x+y) + \tan(x-y) = \frac{2 \tan x \sec^2 y}{1 - \tan^2 x \tan^2 y}.$$

$$251. \sin(x+y) + \cos(x-y) = (\sin x + \cos x)(\sin y + \cos y).$$

$$252. \tan \frac{1}{2}(x-y) = \frac{\sin x - \sin y}{\cos x + \cos y} = \frac{\cos y - \cos x}{\sin y + \sin x}.$$

$$253. \tan(45^\circ + x) = \frac{\cos x + \sin x}{\cos x - \sin x} = \frac{1 + \tan x}{1 - \tan x}.$$

$$254. \frac{\tan x}{\tan y} = \frac{\sin(x+y) + \sin(x-y)}{\sin(x+y) - \sin(x-y)}.$$

$$255. \tan^2 \frac{1}{2}x + 2 \cot x \tan \frac{1}{2}x - 1 = 0.$$

$$256. \tan^2 \frac{1}{2}x - 2 \csc x \tan \frac{1}{2}x + 1 = 0.$$

Explain how the general quadratic equation may be solved by the use of 255 or 256.

$$257. \tan \frac{1}{2}x = \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x}.$$

$$258. \sin nx = 2 \sin(n-1)x \cos x - \sin(n-2)x.$$

$$259. \cos nx = 2 \cos(n-1)x \cos x - \cos(n-2)x.$$

$$260. \tan nx = \frac{\tan(n-1)x + \tan x}{1 - \tan(n-1)x \tan x}.$$

Solve the following equations :

261.  $a \tan^2 x + b = c \sec x.$

262.  $\cos^2 x - \sin^2 x + \tan^2 x = \frac{5}{6}.$

263.  $\sin(20^\circ + x) \cos(20^\circ - x) = .2.$

264.  $\tan x + \cot x = 4 \cot 2x.$

265.  $\cot x - \tan x = 4 \cos 2x.$

266.  $\tan 2x + \tan 3x = 3 \tan x.$

267.  $1 - \cos 2x = 3 \sin 2x.$

268.  $\sin x + \cos x = \sqrt{2}.$

269.  $16(\cot x - \sec x) = \tan x.$

270.  $\tan x + \cot x = 2 \sec 2x.$

271.  $\cot x \tan 2x - \tan x \cot 2x = 2.$

272.  $\sec^2 x + \tan x = 3.$

273.  $\arctan(N+1) = 3 \arctan(N-1).$

274.  $\arctan \frac{1}{N-1} - \arctan \frac{1}{N+1} = \frac{1}{12} \pi.$

## ANSWERS.



1.  $c \tan A = 102.7$ .
2.  $a \cot 12^\circ 40' = 387.1$ .
3.  $\cot x = 2.4$ ;  $22^\circ 37'$ .
4.  $23 \sec 38^\circ = 29.19$ .
5.  $880 \sin 33^\circ 3' = 480$ .
6.  $c \tan A = 125.1$ .
7.  $c \tan A + d = 18$ .
8.  $c(\tan A + \tan A_1) = 102$ .
9.  $a(1 + \tan A \cot A_1) = 76.5$ .
10.  $\frac{a \tan A_1}{\tan A_1 - \tan A} = 100$ .
11.  $\frac{a}{\tan A + \tan A_1} = 500$ .
12.  $a(\tan C - \tan C_1) = 1.5$ .
13.  $c(\tan A_1 - \tan A) = 128.4$ .
14.  $\frac{c}{\cot A + \cot A_1} = 600$ .
15.  $\frac{c \tan A_1}{\tan A_1 - \tan A_1} = 60$ .
16.  $\sqrt{cc_1} = 150$ .
17.  $\frac{t \tan A}{\tan A_1 - \tan A} = 9$ .
18.  $(a - h) \cot A = 3.1$ .
19.  $\cot x = n$ ;  $51^\circ 41'$ .
20.  $\tan x = n$ ;  $47^\circ 44'$ .
21.  $a\sqrt{\cot^2 A + \cot^2 D} = 109$ .
22.  $\frac{d}{\sqrt{\cot^2 D - \cot^2 A}} = 125$ .
23.  $b^2 \sin 2A = b^2 - (a - c)^2$ ;  
N.  $22^\circ 32'$  E.
24.  $4K = 918(a - b + c)$ ;  
340, 189.
25.  $b^2 = (a + c)^2 - 4K$ ; 377.
26. 13.7, 30.8.
27. 4.844, 8.72.
28. 14.25, 8.75.
29.  $154 \tan 27^\circ 10' + 5 = 84$ .
30. 3034, 52.94.
31.  $48^\circ 50'$ .
32. 12.6,  $54^\circ 52'$ .
33.  $\frac{2\pi R \cos 40^\circ}{86400} = .2206$ .
34.  $2^\circ 18.7'$ , .1615 m., 852.6 ft.
35.  $\tan x = \frac{m+n}{m-n}$ ;  $77^\circ$ .
36.  $\frac{1}{2}b\sqrt{2} \tan 43^\circ 35' = 1700$ .
37.  $a\sqrt{\frac{h+a}{h-a}} = 60$ .
38.  $\frac{2.5}{27} \csc 33^\circ 45' = 10$ .

39.  $13.5 \csc 58^\circ 6' = 15.9$ .
40.  $\frac{100}{\cot 20^\circ 15' + 2 \cos 20^\circ 15'} = 21.8$ .
41.  $\sec A = 2.4$ ;  $A = 65^\circ 23'$ ;  $K = 10.64$ .
42.  $1.5 d^2 \sin 60^\circ \sin^2 \frac{1}{2} A = 8.798$ .
43. Area = two triangles - sector =  $\frac{1}{2} x^2 \sin 60^\circ$ ; 15.
44.  $\pi b^2 \cos A = 204.2$ .      45.  $\pi K \sec \frac{1}{2} C = 33.49$ .
46.  $36 \pi V^2 = S^3 \sin^2 C \csc \frac{1}{2} C$ ; 73.43.
47. rad base =  $1.81 \csc 54^\circ 18'$ ;  $V = 36.65$ .
48.  $p_1 \sec A = 2p$ ;  $K = p^2 \cot A = 25.57$ .
49.  $\tan A = \frac{1}{2} \pi$ ;  $A = 57^\circ 31'$ .    57.  $2.143 \pi \cot 25^\circ 43' = 14$ .
50.  $100^\circ$ .    51. 50.    52.  $48^\circ 55'$ .    58.  $\begin{cases} \frac{1}{2} a \csc \frac{90^\circ}{2n+1} = 8.988. \\ 2a \cos \frac{180^\circ}{2n+1} = 7.208. \end{cases}$
53.  $\frac{106.27 \pi b}{180 a} = 6.185$ .
54. 21.    55.  $163^\circ 34'$ .
56.  $.3125 d^2 \csc^2 36^\circ = 12.15$ .    59. 143.1 feet.
60.  $27 r^2 \tan 6^\circ 40' \sin^2 6^\circ 40' = 3.6$ .
61.  $2 \frac{m}{n} K \csc \frac{360^\circ}{n} \tan \frac{180^\circ}{m} = 116.5$ .
62.  $\frac{1}{2} r^2 (16.96 \pi + \sin 70^\circ 32' - \sin 40^\circ) = .4656$ .
63.  $441 \cos 46^\circ \tan 20^\circ = 157$ .    67.  $c \sin (A_1 - A) \sec A \sec A_1$ .
64.  $a \csc (A_1 - A) \cos A \sin A_1$ .    68.  $c \csc (A + A_1) \sin A \sin A_1$ .
65.  $a \csc (A + A_1) \cos A \cos A_1$ .    69.  $c \csc (A - A_1) \cos A \sin A_1$ .
66.  $a \sin (C - C_1) \sec C \sec C_1$ .    70. Apply (19); 890.
71. Apply (20); 741.2.
72.  $18 \sec 87^\circ 30' \cos 37^\circ 30' \sin 40^\circ = 210$ .
73.  $50 \csc 5^\circ 55' \cos 31^\circ 12' \sin 25^\circ 17' = 177.1$ .  
 $50 \csc 5^\circ 55' \cos 25^\circ 17' = 438.4$ .
74. Apply (24); N.  $87^\circ 49'$  E.  
 Apply (19); 12 miles an hour.

75.  $528 \csc 37^\circ \sin 42^\circ \sec 29^\circ \sin 8^\circ = 114.5$ .  
 $528 \csc 37^\circ \sin 5^\circ = 301$ .
76.  $c \csc C \sin A \sin B - 50 = 240$ .
77.  $775.3 \csc 27^\circ 35' \sin 126^\circ 43' \sin 32^\circ 57' = 730$ .
78.  $200 \sec^2 26^\circ 37' \csc 10^\circ 8' \sin 10^\circ 22' \sin 20^\circ 30' = 89.63$ .
79.  $180 \csc 17^\circ 15' \sin 33^\circ 45' \sec 40^\circ \sin 11^\circ = 84$ .
80. Apply (27);  $18.71 + 16.2 = 34.91$ .
81. Apply (24);  $\cot x = 1.76 \csc 30^\circ - \cot 30^\circ$ .  
 $29^\circ 15'$  with the front of the train.
82.  $6 \sin 45^\circ \csc 10^\circ = 24.44$ .      85. 3.5.
83. N.  $70^\circ 44'$  E.; 9.983.      86. Apply (30); 229,000.
84. 10.32;  $78^\circ 14'$ .      87. Apply (29); 52.
88. Apply (26);  $b = x \cot A$ ,  $l = x \cot D$ ,  $a = x \cot B$ ; 60.
89. Apply (19); 20.51; 23.63.
90. Apply (16), (19), and (20);  

$$\cos A = \frac{c+4}{2(c+1)} = \frac{c+1}{2(c-1)}; c = 5.$$
91.  $c \cos \frac{1}{4}(B+A) \sin \frac{1}{4}(B-A) = 57$ .
92. Snell's; 1034, 1465, 2047.
93.  $300 \csc 68^\circ 14' \sin 120^\circ 26' \csc 17^\circ 48' \sin 70^\circ = 856.2$ .
94.  $100 \sin 88^\circ 10' \csc 8^\circ 20' = 676$ .      95. Apply (25); 5.715.
96. Apply (25); 331.      97. Apply (28); 1054.
98. Find  $AB$  and  $BD$  and apply (20); 287.2.
99. Find  $AB$ , then  $BC = .3537$ .
100.  $\sin ADB = \tan 56^\circ 26' \cot 65^\circ 23' \sin 56^\circ 33'$ ; 114.
101. Apply (19); 410.
102. Same as Problem 88; 100.
103. Snell's; 229, 61, 109.
104. By (30)  $\cos^2 A = .85$ ;  $A = 22^\circ 47'$ .  
 By (28)  $\cot x = 1.125 \csc 45^\circ 34' - \cot 45^\circ 34'$ ;  $x = 59^\circ 15'$ .  
 By (19)  $y = 9 \csc 22^\circ 47' \sin 82^\circ 2' \cos 59^\circ 15' = 11.77$ .

105.  $\frac{1}{2} (12 \csc A) (16 \csc A) \sin A = 96 \csc A = 100.$

106. Divide into a triangle and a parallelogram; 46.82.

107.  $\frac{1}{2} a \frac{v+w \pm \sqrt{3vw}}{v^2+w^2-vw} = 9 \text{ sec. or } 1 \text{ min. } 57 \text{ sec.}$

108.  $70 \sin \frac{1}{2} (A_1 - A_2) \sin \frac{1}{2} (B_1 + B_2) \times$   
 $\csc \frac{1}{2} (A_1 + A_2 + B_1 + B_2) = 18.34.$

109. The pond is the incircle of the triangle formed by the three tangents; 36.58.

110. The pond is the excircle of the triangle formed by the three tangents; 768.3.

111. The hill is at the center of the circumcircle.

$60.8 \csc 15^\circ 25' \tan 52^\circ 40' = 300.$

112.  $r \tan \frac{1}{2} B \tan \frac{1}{2} C = 1.589.$     113.  $r \tan^2 (45 - \frac{1}{4} A) = 1.718.$

114.  $4r \sin (45^\circ - \frac{1}{4} A) \sin (45^\circ - \frac{1}{4} B) \sin (45^\circ - \frac{1}{4} C) = 2.925.$

115. 3107.

116. 5057.

117. 2713.

118. 2410 geographical miles.

119. 7 hours.

120. lat.  $37^\circ 36'$ ; long.  $68^\circ 38'$ .

121.  $\cos Z = -\sec l \sin d.$      $238^\circ 42'$ ;  $270^\circ$ ;  $301^\circ 18'$ ;  $180^\circ.$

$\cos P = -\tan l \tan d.$     length =  $2P \div 15.$

14 hr. 50 min. 48 sec.; 12 hr.; 9 hr. 9 min. 12 sec.; 24 hr.

122.  $\sec P = \tan l \cot d.$     3 hr. 55 min. 28 sec.; 6; 12 P.M.

123.  $\sin a = \sin l \sin d.$      $14^\circ 49'.$      $\tan Z = \sec l \cot d.$      $251^\circ 37'.$

124.  $\tan l = \tan d \sec 31^\circ 30';$   $14^\circ 46'.$

125. Let  $R$  be first position,  $S$  the second.

$RS = 49^\circ 40',$   $PRS = 84^\circ 2',$   $ZRS = 50^\circ 31';$  lat. =  $47^\circ 55'.$

126.  $P = 32^\circ 30',$   $ZS = 45^\circ 40',$   $PS = 116^\circ 6';$  lat. =  $6^\circ 56'.$

127.  $40^\circ 6'.$

128.  $.336 R^2.$

129.  $2.443 R^2.$

130.  $\frac{f \log d - c \log a}{b \log a - e \log d} = -1.121.$     131.  $\frac{\log e}{b \log a + d \log c} = .216.$

132. Roots of  $x^2 \log d - x \log b = \log a - \log c.$  Imaginary.

$$133. x = \frac{\log b (f \log e - d \log c)}{\log a \log e - \log b \log c} = -21.93.$$

$$134. x = \frac{1}{2} \left[ \frac{\log b}{\log a} + \frac{\log a}{\log b} \right] = 1.108.$$

135. Roots of  $x^2 \log a + x \log b + \log c = 0$ . Imaginary.

$$136. \frac{\log \log c - \log \log a}{\log b} = .6309.$$

$$137. \tan x = \frac{.6 \sin 20^\circ - \sin 10^\circ}{\cos 10^\circ - .6 \cos 20^\circ}; 2^\circ 56'.$$

138. Reduce to sines and cosines and apply (69),  
 $\sin (2x - 10^\circ) = -5 \sin 10^\circ; 125^\circ 7'.$

139. Apply the value of  $\tan (x \pm y)$  and reduce  
 $\cot^2 x = 1 + 2 \tan^2 10^\circ; 44^\circ 8'.$

140. Apply (65) and (75);  $\cot x = -(2 + \cot 10^\circ); 172^\circ 34'.$

141.  $\cos (2x - 10^\circ) = 1.2 - \cos 10^\circ; 43^\circ 47'.$

142. By (13);  $32 \cos^2 x = 16 + 16 \cos 2x$ .  
 $16 \cos^2 2x + 16 \cos 2x + 3 = 0; x = 52^\circ 14'.$

143. Apply (69);  $29^\circ 33'$ . 144.  $4 \cos x = \sqrt{17} - 1; 38^\circ 40'.$

145.  $\sin (2x - 70^\circ) = .6 \csc 70^\circ; 105^\circ 10'.$

146.  $\tan x = .4142; 22^\circ 30'$ . 147.  $\tan x = \tan a \tan b \tan c.$

148.  $160 \tan A = 70 \tan (A + 21^\circ) = 168.$

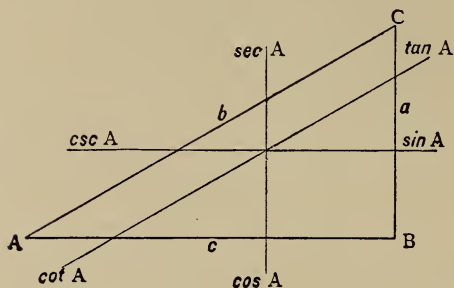
149.  $a \tan A = (a - b) \tan 3A = 59.2.$

150.  $97^\circ 37'$  with the road; 11.44 miles.

151.  $360 \cot A = 360 \cot (A - 2^\circ 7') = 4994.$

152.  $\frac{1}{2} r^2 (\theta - \sin \theta) = 13.714$  and  $4.252; 9.462.$   
 $\pi r^2 - (13.714 + 4.252) = 1239.$

153. Two segments;  $\theta_1 = 153^\circ 44', \theta_2 = 54^\circ 33'; .1 A.$



## PLANE RIGHT TRIANGLES.

Any side = adjacent side times adjacent function.

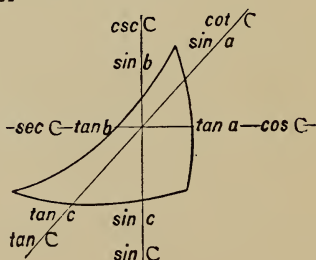
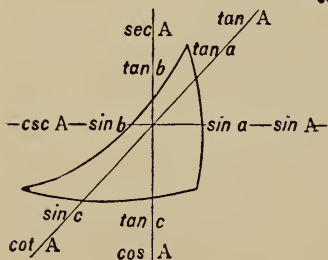
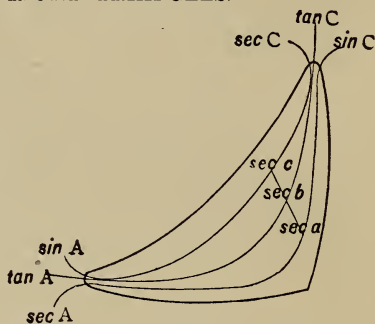
Any function of  $A$  = adjacent side divided by the next side.

Any function of  $C$  = the corresponding cofunction of  $A$ .

## SPHERICAL RIGHT TRIANGLES.

Any function is equal to the product of its adjacent functions.

Any function is equal to the product of an adjacent function and the reciprocal of the next function.



A TABLE OF  
FOUR-PLACE MANTISSÆ OF LOGARITHMS  
OF NUMBERS

FROM 1 TO 1000

ALSO

A TABLE OF  
THE LOGARITHMS OF THE TRIGONOMETRIC  
FUNCTIONS OF ANGLES

FROM  $0^{\circ}$  TO  $90^{\circ}$  FOR EVERY  $10'$

ALSO

A TABLE OF FORMULÆ

ALSO

A TABLE OF VALUES OF CONSTANTS

| N  | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 10 | .0000 | .0043 | .0086 | .0128 | .0170 | .0212 | .0253 | .0294 | .0334 | .0374 |
| 11 | 414   | 453   | 492   | 531   | 569   | 607   | 645   | 682   | 719   | 755   |
| 12 | 792   | 828   | 864   | 899   | 934   | 969   | .1004 | .1038 | .1072 | .1106 |
| 13 | .1139 | .1173 | .1206 | .1239 | .1271 | .1303 | 335   | 367   | 399   | 430   |
| 14 | 461   | 492   | 523   | 553   | 584   | 614   | 644   | 673   | 703   | 732   |
| 15 | .1761 | .1790 | .1818 | .1847 | .1875 | .1903 | .1931 | .1959 | .1987 | .2014 |
| 16 | .2041 | 2068  | .2095 | .2122 | .2148 | .2175 | .2201 | .2227 | .2253 | 279   |
| 17 | 304   | 330   | 355   | 380   | 405   | 430   | 455   | 480   | 504   | 529   |
| 18 | 553   | 577   | 601   | 625   | 648   | 672   | 695   | 718   | 742   | 765   |
| 19 | 788   | 810   | 833   | 856   | 878   | 900   | 923   | 945   | 967   | 989   |
| 20 | .3010 | .3032 | .3054 | .3075 | .3096 | .3118 | .3139 | .3160 | .3181 | .3201 |
| 21 | 222   | 243   | 263   | 284   | 304   | 324   | 345   | 365   | 385   | 404   |
| 22 | 424   | 444   | 464   | 483   | 502   | 522   | 541   | 560   | 579   | 598   |
| 23 | 617   | 636   | 655   | 674   | 692   | 711   | 729   | 747   | 766   | 784   |
| 24 | 802   | 820   | 838   | 856   | 874   | 892   | 909   | 927   | 945   | 962   |
| 25 | .3979 | .3997 | .4014 | .4031 | .4048 | .4065 | .4082 | .4099 | .4116 | .4133 |
| 26 | .4150 | .4166 | 183   | 200   | 216   | 232   | 249   | 265   | 281   | 298   |
| 27 | 314   | 330   | 346   | 362   | 378   | 393   | 409   | 425   | 440   | 456   |
| 28 | 472   | 487   | 502   | 518   | 533   | 548   | 564   | 579   | 594   | 609   |
| 29 | 624   | 639   | 654   | 669   | 683   | 698   | 713   | 728   | 742   | 757   |
| 30 | .4771 | .4786 | .4800 | .4814 | .4829 | .4843 | .4857 | .4871 | .4886 | .4900 |
| 31 | 914   | 928   | 942   | 955   | 969   | 983   | 997   | .5011 | .5024 | .5038 |
| 32 | .5051 | .5065 | .5079 | .5092 | .5105 | .5119 | .5132 | 145   | 159   | 172   |
| 33 | 185   | 198   | 211   | 224   | 237   | 250   | 263   | 276   | 289   | 302   |
| 34 | 315   | 328   | 340   | 353   | 366   | 378   | 391   | 403   | 416   | 428   |
| 35 | .5441 | .5453 | .5465 | .5478 | .5490 | .5502 | .5514 | .5527 | .5539 | .5551 |
| 36 | 563   | 575   | 587   | 599   | 611   | 623   | 635   | 647   | 658   | 670   |
| 37 | 682   | 694   | 705   | 717   | 729   | 740   | 752   | 763   | 775   | 786   |
| 38 | 798   | 809   | 821   | 832   | 843   | 855   | 866   | 877   | 888   | 899   |
| 39 | 911   | 922   | 933   | 944   | 955   | 966   | 977   | 988   | 999   | .6010 |
| 40 | .6021 | .6031 | .6042 | .6053 | .6064 | .6075 | .6085 | .6096 | .6107 | .6117 |
| 41 | 128   | 138   | 149   | 160   | 170   | 180   | 191   | 201   | 212   | 222   |
| 42 | 232   | 243   | 253   | 263   | 274   | 284   | 294   | 304   | 314   | 325   |
| 43 | 335   | 345   | 355   | 365   | 375   | 385   | 395   | 405   | 415   | 425   |
| 44 | 435   | 444   | 454   | 464   | 474   | 484   | 493   | 503   | 513   | 522   |
| 45 | .6532 | .6542 | .6551 | .6561 | .6571 | .6580 | .6590 | .6599 | .6609 | .6618 |
| 46 | 628   | 637   | 646   | 656   | 665   | 675   | 684   | 693   | 702   | 712   |
| 47 | 721   | 730   | 739   | 749   | 758   | 767   | 776   | 785   | 794   | 803   |
| 48 | 812   | 821   | 830   | 839   | 848   | 857   | 866   | 875   | 884   | 893   |
| 49 | 902   | 911   | 920   | 928   | 937   | 946   | 955   | 964   | 972   | 981   |
| 50 | .6990 | .6998 | .7007 | .7016 | .7024 | .7033 | .7042 | .7050 | .7059 | .7067 |
| 51 | .7076 | .7084 | 093   | 101   | 110   | 118   | 126   | 135   | 143   | 152   |
| 52 | 160   | 168   | 177   | 185   | 193   | 202   | 210   | 218   | 226   | 235   |
| 53 | 243   | 251   | 259   | 267   | 275   | 284   | 292   | 300   | 308   | 316   |
| 54 | 324   | 332   | 340   | 348   | 356   | 364   | 372   | 380   | 388   | 396   |

| N         | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| <b>55</b> | .7404 | .7412 | .7419 | .7427 | .7435 | .7443 | .7451 | .7459 | .7466 | .7474 |
| 56        | 482   | 490   | 497   | 505   | 513   | 520   | 528   | 536   | 543   | 551   |
| 57        | 559   | 566   | 574   | 582   | 589   | 597   | 604   | 612   | 619   | 627   |
| 58        | 634   | 642   | 649   | 657   | 664   | 672   | 679   | 686   | 694   | 701   |
| 59        | 709   | 716   | 723   | 731   | 738   | 745   | 752   | 760   | 767   | 774   |
| <b>60</b> | .7782 | .7789 | .7796 | .7803 | .7810 | .7818 | .7825 | .7832 | .7839 | .7846 |
| 61        | 853   | 860   | 868   | 875   | 882   | 889   | 896   | 903   | 910   | 917   |
| 62        | 924   | 931   | 938   | 945   | 952   | 959   | 966   | 973   | 980   | 987   |
| 63        | 993   | .8000 | .8007 | .8014 | .8021 | .8028 | .8035 | .8041 | .8048 | .8055 |
| 64        | .8062 | 069   | 075   | 082   | 089   | 096   | 102   | 109   | 116   | 122   |
| <b>65</b> | .8129 | .8136 | .8142 | .8149 | .8156 | .8162 | .8169 | .8176 | .8182 | .8189 |
| 66        | 195   | 202   | 209   | 215   | 222   | 228   | 235   | 241   | 248   | 254   |
| 67        | 261   | 267   | 274   | 280   | 287   | 293   | 299   | 306   | 312   | 319   |
| 68        | 325   | 331   | 338   | 344   | 351   | 357   | 363   | 370   | 376   | 382   |
| 69        | 388   | 395   | 401   | 407   | 414   | 420   | 426   | 432   | 439   | 445   |
| <b>70</b> | .8451 | .8457 | .8463 | .8470 | .8476 | .8482 | .8488 | .8494 | .8500 | .8506 |
| 71        | 513   | 519   | 525   | 531   | 537   | 543   | 549   | 555   | 561   | 567   |
| 72        | 573   | 579   | 585   | 591   | 597   | 603   | 609   | 615   | 621   | 627   |
| 73        | 633   | 639   | 645   | 651   | 657   | 663   | 669   | 675   | 681   | 686   |
| 74        | 692   | 698   | 704   | 710   | 716   | 722   | 727   | 733   | 739   | 745   |
| <b>75</b> | .8751 | .8756 | .8762 | .8768 | .8774 | .8779 | .8785 | .8791 | .8797 | .8802 |
| 76        | 808   | 814   | 820   | 825   | 831   | 837   | 842   | 848   | 854   | 859   |
| 77        | 865   | 871   | 876   | 882   | 887   | 893   | 899   | 904   | 910   | 915   |
| 78        | 921   | 927   | 932   | 938   | 943   | 949   | 954   | 960   | 965   | 971   |
| 79        | 976   | 982   | 987   | 993   | 998   | .9004 | .9009 | .9015 | .9020 | .9025 |
| <b>80</b> | .9031 | .9036 | .9042 | .9047 | .9053 | .9058 | .9063 | .9069 | .9074 | .9079 |
| 81        | 085   | 090   | 096   | 101   | 106   | 112   | 117   | 122   | 128   | 133   |
| 82        | 138   | 143   | 149   | 154   | 159   | 165   | 170   | 175   | 180   | 186   |
| 83        | 191   | 196   | 201   | 206   | 212   | 217   | 222   | 227   | 232   | 238   |
| 84        | 243   | 248   | 253   | 258   | 263   | 269   | 274   | 279   | 284   | 289   |
| <b>85</b> | .9294 | .9299 | .9304 | .9309 | .9315 | .9320 | .9325 | .9330 | .9335 | .9340 |
| 86        | 345   | 350   | 355   | 360   | 365   | 370   | 375   | 380   | 385   | 390   |
| 87        | 395   | 400   | 405   | 410   | 415   | 420   | 425   | 430   | 435   | 440   |
| 88        | 445   | 450   | 455   | 460   | 465   | 469   | 474   | 479   | 484   | 489   |
| 89        | 494   | 499   | 504   | 509   | 513   | 518   | 523   | 528   | 533   | 538   |
| <b>90</b> | .9542 | .9547 | .9552 | .9557 | .9562 | .9566 | .9571 | .9576 | .9581 | .9586 |
| 91        | 590   | 595   | 600   | 605   | 609   | 614   | 619   | 624   | 628   | 633   |
| 92        | 638   | 643   | 647   | 652   | 657   | 661   | 666   | 671   | 675   | 680   |
| 93        | 685   | 689   | 694   | 699   | 703   | 708   | 713   | 717   | 722   | 727   |
| 94        | 731   | 736   | 741   | 745   | 750   | 754   | 759   | 763   | 768   | 773   |
| <b>95</b> | .9777 | .9782 | .9786 | .9791 | .9795 | .9800 | .9805 | .9809 | .9814 | .9818 |
| 96        | 823   | 827   | 832   | 836   | 841   | 845   | 850   | 854   | 859   | 863   |
| 97        | 868   | 872   | 877   | 881   | 886   | 890   | 894   | 899   | 903   | 908   |
| 98        | 912   | 917   | 921   | 926   | 930   | 934   | 939   | 943   | 948   | 952   |
| 99        | 956   | 961   | 965   | 969   | 974   | 978   | 983   | 987   | 991   | 996   |

| $x$       | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|-----------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>0°</b> | Neg. Inf.  | Neg. Inf.  | 0.000 0       | Infinite      | Infinite      | 10.000 0   | <b>90°</b> |
| 10'       | 7. 4637    | 7. 4637    | 0             | 2. 5363       | 2. 5363       | 0          | 50'        |
| 20'       | 7648       | 7648       | 0             | 2352          | 2352          | 0          | 40'        |
| 30'       | 9408       | 9409       | 0             | 0592          | 0591          | 0          | 30'        |
| 40'       | 8. 0658    | 8. 0658    | 0             | 1. 9342       | 1. 9342       | 0          | 20'        |
| 50'       | 1627       | 1627       | 0             | 8373          | 8373          | 0          | 10'        |
| <b>1°</b> | 8. 2419    | 8. 2419    | 0.000 1       | 1. 7581       | 1. 7581       | 9.999 9    | <b>89°</b> |
| 10'       | 3088       | 3089       | 1             | 6912          | 6911          | 9          | 50'        |
| 20'       | 3668       | 3669       | 1             | 6332          | 6331          | 9          | 40'        |
| 30'       | 4179       | 4181       | 1             | 5821          | 5819          | 9          | 30'        |
| 40'       | 4637       | 4638       | 2             | 5363          | 5362          | 8          | 20'        |
| 50'       | 5050       | 5053       | 2             | 4950          | 4947          | 8          | 10'        |
| <b>2°</b> | 8. 5428    | 8. 5431    | 0.000 3       | 1. 4572       | 1. 4569       | 9.999 7    | <b>88°</b> |
| 10'       | 5776       | 5779       | 3             | 4224          | 4221          | 7          | 50'        |
| 20'       | 6097       | 6101       | 4             | 3903          | 3899          | 6          | 40'        |
| 30'       | 6397       | 6401       | 4             | 3603          | 3599          | 6          | 30'        |
| 40'       | 6677       | 6682       | 5             | 3323          | 3318          | 5          | 20'        |
| 50'       | 6940       | 6945       | 5             | 3060          | 3055          | 5          | 10'        |
| <b>3°</b> | 8. 7188    | 8. 7194    | 0.000 6       | 1. 2812       | 1. 2806       | 9.999 4    | <b>87°</b> |
| 10'       | 7423       | 7429       | 7             | 2577          | 2571          | 3          | 50'        |
| 20'       | 7645       | 7652       | 7             | 2355          | 2348          | 3          | 40'        |
| 30'       | 7857       | 7865       | 8             | 2143          | 2135          | 2          | 30'        |
| 40'       | 8059       | 8067       | 9             | 1941          | 1933          | 1          | 20'        |
| 50'       | 8251       | 8261       | 0.001 0       | 1749          | 1739          | 0          | 10'        |
| <b>4°</b> | 8. 8436    | 8. 8446    | 0.001 1       | 1. 1564       | 1. 1554       | 9.998 9    | <b>86°</b> |
| 10'       | 8613       | 8624       | 1             | 1387          | 1376          | 9          | 50'        |
| 20'       | 8783       | 8795       | 2             | 1217          | 1205          | 8          | 40'        |
| 30'       | 8946       | 8960       | 3             | 1054          | 1040          | 7          | 30'        |
| 40'       | 8. 9104    | 8. 9118    | 4             | 1. 0896       | 1. 0882       | 6          | 20'        |
| 50'       | 9256       | 9272       | 5             | 0744          | 0728          | 5          | 10'        |
| <b>5°</b> | 8. 9403    | 8. 9420    | 0.001 7       | 1. 0597       | 1. 0580       | 9.998 3    | <b>85°</b> |
| 10'       | 9545       | 9563       | 8             | 0455          | 0437          | 2          | 50'        |
| 20'       | 9682       | 9701       | 9             | 0318          | 0299          | 1          | 40'        |
| 30'       | 9816       | 9836       | 0.002 0       | 0184          | 0164          | 0          | 30'        |
| 40'       | 9945       | 9966       | 1             | 0055          | 0034          | 9.997 9    | 20'        |
| 50'       | 9. 0070    | 9. 0093    | 3             | 0. 9930       | 0. 9907       | 7          | 10'        |
| <b>6°</b> | 9. 0192    | 9. 0216    | 0.002 4       | 0. 9808       | 0. 9784       | 9.997 6    | <b>84°</b> |
| 10'       | 0311       | 0336       | 5             | 9689          | 9664          | 5          | 50'        |
| 20'       | 0426       | 0453       | 7             | 9574          | 9547          | 3          | 40'        |
| 30'       | 0539       | 0567       | 8             | 9461          | 9433          | 2          | 30'        |
| 40'       | 0648       | 0678       | 9             | 9352          | 9322          | 1          | 20'        |
| 50'       | 0755       | 0786       | 0.003 1       | 9245          | 9214          | 9.996 9    | 10'        |
| <b>7°</b> | 0859       | 9. 0891    | 0.003 2       | 9141          | 0. 9109       | 9.996 8    | <b>83°</b> |
|           | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $L \cot x$    | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>7°</b>  | 9.0 859    | 9.0 891    | 0.00 32       | 0.9 141       | 0.9 109       | 9.99 68    | <b>83°</b> |
| 10'        | 961        | 995        | 34            | 039           | 005           | 66         | 50'        |
| 20'        | 9.1 060    | 9.1 096    | 36            | 0.8 940       | 0.8 904       | 64         | 40'        |
| 30'        | 157        | 194        | 37            | 843           | 806           | 63         | 30'        |
| 40'        | 252        | 291        | 39            | 748           | 709           | 61         | 20'        |
| 50'        | 345        | 385        | 0.00 41       | 665           | 615           | 59         | 10'        |
| <b>8°</b>  | 9.1 436    | 9.1 478    | 0.00 42       | 0.8 564       | 0.8 522       | 9.99 58    | <b>82°</b> |
| 10'        | 525        | 569        | 44            | 475           | 431           | 56         | 50'        |
| 20'        | 612        | 658        | 46            | 388           | 342           | 54         | 40'        |
| 30'        | 697        | 745        | 48            | 303           | 255           | 52         | 30'        |
| 40'        | 781        | 831        | 50            | 219           | 169           | 50         | 20'        |
| 50'        | 863        | 915        | 52            | 137           | 085           | 48         | 10'        |
| <b>9°</b>  | 9.1 943    | 9.1 997    | 0.00 54       | 0.8 057       | 0.8 003       | 9.99 46    | <b>81°</b> |
| 10'        | 9.2 022    | 9.2 078    | 56            | 0.7 978       | 0.7 922       | 44         | 50'        |
| 20'        | 100        | 158        | 58            | 900           | 842           | 42         | 40'        |
| 30'        | 176        | 236        | 60            | 824           | 764           | 40         | 30'        |
| 40'        | 251        | 313        | 62            | 749           | 687           | 38         | 20'        |
| 50'        | 324        | 389        | 64            | 676           | 611           | 36         | 10'        |
| <b>10°</b> | 9.2 397    | 9.2 463    | 0.00 66       | 0.7 603       | 0.7 537       | 9.99 34    | <b>80°</b> |
| 10'        | 468        | 536        | 69            | 532           | 464           | 31         | 50'        |
| 20'        | 538        | 609        | 71            | 462           | 391           | 29         | 40'        |
| 30'        | 606        | 680        | 73            | 394           | 320           | 27         | 30'        |
| 40'        | 674        | 750        | 76            | 326           | 250           | 24         | 20'        |
| 50'        | 740        | 819        | 78            | 260           | 181           | 22         | 10'        |
| <b>11°</b> | 9.2 806    | 9.2 887    | 0.00 81       | 0.7 194       | 0.7 113       | 9.99 19    | <b>79°</b> |
| 10'        | 870        | 953        | 83            | 130           | 047           | 17         | 50'        |
| 20'        | 934        | 9.3 020    | 86            | 066           | 0.6 980       | 14         | 40'        |
| 30'        | 997        | 085        | 88            | 003           | 915           | 12         | 30'        |
| 40'        | 9.3 058    | 149        | 91            | 0.6 942       | 851           | 09         | 20'        |
| 50'        | 119        | 212        | 93            | 881           | 788           | 07         | 10'        |
| <b>12°</b> | 9.3 179    | 9.3 275    | 0.00 96       | 0.6 821       | 0.6 725       | 9.99 04    | <b>78°</b> |
| 10'        | 238        | 336        | 99            | 762           | 664           | 01         | 50'        |
| 20'        | 296        | 397        | 0.01 01       | 704           | 603           | 9.98 99    | 40'        |
| 30'        | 353        | 458        | 04            | 647           | 542           | 96         | 30'        |
| 40'        | 410        | 517        | 07            | 590           | 483           | 93         | 20'        |
| 50'        | 466        | 576        | 10            | 534           | 424           | 90         | 10'        |
| <b>13°</b> | 9.3 521    | 9.3 634    | 0.01 13       | 0.6 479       | 0.6 366       | 9.98 87    | <b>77°</b> |
| 10'        | 575        | 691        | 16            | 425           | 309           | 84         | 50'        |
| 20'        | 629        | 748        | 19            | 371           | 252           | 81         | 40'        |
| 30'        | 682        | 804        | 22            | 318           | 196           | 78         | 30'        |
| 40'        | 734        | 859        | 25            | 266           | 141           | 75         | 20'        |
| 50'        | 786        | 914        | 28            | 214           | 086           | 72         | 10'        |
| <b>14°</b> | 9.3 837    | 9.3 968    | 0.01 31       | 0.6 163       | 0.6 032       | 9.98 69    | <b>76°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>14°</b> | 9.3 837    | 9.3 968    | 0.01 31       | 0.6 163       | 0.6 032       | 9.98 69    | <b>76°</b> |
| 10'        | 887        | 9.4 021    | 34            | 113           | 0.5 979       | 66         | 50'        |
| 20'        | 937        | 074        | 37            | 063           | 926           | 63         | 40'        |
| 30'        | 986        | 127        | 41            | 014           | 873           | 59         | 30'        |
| 40'        | 9.4 035    | 178        | 44            | 0.5 965       | 822           | 56         | 20'        |
| 50'        | 083        | 230        | 47            | 917           | 770           | 53         | 10'        |
| <b>15°</b> | 9.4 130    | 9.4 281    | 0.01 51       | 0.5 870       | 0.5 719       | 9.98 49    | <b>75°</b> |
| 10'        | 177        | 331        | 54            | 823           | 669           | 46         | 50'        |
| 20'        | 223        | 381        | 57            | 777           | 619           | 43         | 40'        |
| 30'        | 269        | 430        | 61            | 731           | 570           | 39         | 30'        |
| 40'        | 314        | 479        | 64            | 686           | 521           | 36         | 20'        |
| 50'        | 359        | 527        | 68            | 641           | 473           | 32         | 10'        |
| <b>16°</b> | 9.4 403    | 9.4 575    | 0.01 72       | 0.5 597       | 0.5 425       | 9.98 28    | <b>74°</b> |
| 10'        | 447        | 622        | 75            | 553           | 378           | 25         | 50'        |
| 20'        | 491        | 669        | 79            | 509           | 331           | 21         | 40'        |
| 30'        | 533        | 716        | 83            | 467           | 284           | 17         | 30'        |
| 40'        | 576        | 762        | 86            | 424           | 238           | 14         | 20'        |
| 50'        | 618        | 808        | 90            | 382           | 192           | 10         | 10'        |
| <b>17°</b> | 9.4 659    | 9.4 853    | 0.01 94       | 0.5 341       | 0.5 147       | 9.98 06    | <b>73°</b> |
| 10'        | 700        | 898        | 98            | 300           | 102           | 02         | 50'        |
| 20'        | 741        | 943        | 0.02 02       | 259           | 057           | 9.97 98    | 40'        |
| 30'        | 781        | 987        | 06            | 219           | 013           | 94         | 30'        |
| 40'        | 821        | 9.5 031    | 10            | 179           | 0.4 969       | 90         | 20'        |
| 50'        | 861        | 075        | 14            | 139           | 925           | 86         | 10'        |
| <b>18°</b> | 9.4 900    | 9.5 118    | 0.02 18       | 0.5 100       | 0.4 882       | 9.97 82    | <b>72°</b> |
| 10'        | 939        | 161        | 22            | 061           | 839           | 78         | 50'        |
| 20'        | 977        | 203        | 26            | 023           | 797           | 74         | 40'        |
| 30'        | 9.5 015    | 245        | 30            | 0.4 985       | 755           | 70         | 30'        |
| 40'        | 052        | 287        | 35            | 948           | 713           | 65         | 20'        |
| 50'        | 090        | 329        | 39            | 910           | 671           | 61         | 10'        |
| <b>19°</b> | 9.5 126    | 9.5 370    | 0.02 43       | 0.4 874       | 0.4 630       | 9.97 57    | <b>71°</b> |
| 10'        | 163        | 411        | 48            | 837           | 589           | 52         | 50'        |
| 20'        | 199        | 451        | 52            | 801           | 549           | 48         | 40'        |
| 30'        | 235        | 491        | 57            | 765           | 509           | 43         | 30'        |
| 40'        | 270        | 531        | 61            | 730           | 469           | 39         | 20'        |
| 50'        | 306        | 571        | 66            | 694           | 429           | 34         | 10'        |
| <b>20°</b> | 9.5 341    | 9.5 611    | 0.02 70       | 0.4 659       | 0.4 389       | 9.97 30    | <b>70°</b> |
| 10'        | 375        | 650        | 75            | 625           | 350           | 25         | 50'        |
| 20'        | 409        | 689        | 79            | 591           | 311           | 21         | 40'        |
| 30'        | 443        | 727        | 84            | 557           | 273           | 16         | 30'        |
| 40'        | 477        | 766        | 89            | 523           | 234           | 11         | 20'        |
| 50'        | 510        | 804        | 94            | 490           | 196           | 06         | 10'        |
| <b>21°</b> | 9.5 543    | 9.5 842    | 0.02 98       | 0.4 457       | 0.4 158       | 9.97 02    | <b>69°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>21°</b> | 9.5 543    | 9.5 842    | 0.02 98       | 0.4 457       | 0.4 158       | 9.97 02    | <b>69°</b> |
| 10'        | 576        | 879        | 0.03 03       | 424           | 121           | 9.96 97    | 50'        |
| 20'        | 609        | 917        | 08            | 391           | 083           | 92         | 40'        |
| 30'        | 641        | 954        | 13            | 359           | 046           | 87         | 30'        |
| 40'        | 673        | 991        | 18            | 327           | 009           | 82         | 20'        |
| 50'        | 704        | 9.6 028    | 23            | 296           | 0.3 972       | 77         | 10'        |
| <b>22°</b> | 9.5 736    | 9.6 064    | 0.03 28       | 0.4 264       | 0.3 936       | 9.96 72    | <b>68°</b> |
| 10'        | 767        | 100        | 33            | 233           | 900           | 67         | 50'        |
| 20'        | 798        | 136        | 39            | 202           | 864           | 61         | 40'        |
| 30'        | 828        | 172        | 44            | 172           | 828           | 56         | 30'        |
| 40'        | 859        | 208        | 49            | 141           | 792           | 51         | 20'        |
| 50'        | 889        | 243        | 54            | 111           | 757           | 46         | 10'        |
| <b>23°</b> | 9.5 919    | 9.6 279    | 0.03 60       | 0.4 081       | 0.3 721       | 9.96 40    | <b>67°</b> |
| 10'        | 948        | 314        | 65            | 052           | 686           | 35         | 50'        |
| 20'        | 978        | 348        | 71            | 022           | 652           | 29         | 40'        |
| 30'        | 9.6 007    | 383        | 76            | 0.3 993       | 617           | 24         | 30'        |
| 40'        | 036        | 417        | 82            | 964           | 583           | 18         | 20'        |
| 50'        | 065        | 452        | 87            | 935           | 548           | 13         | 10'        |
| <b>24°</b> | 9.6 093    | 9.6 486    | 0.03 93       | 0.3 907       | 0.3 514       | 9.96 07    | <b>66°</b> |
| 10'        | 121        | 520        | 98            | 879           | 480           | 02         | 50'        |
| 20'        | 149        | 553        | 0.04 04       | 851           | 447           | 9.95 96    | 40'        |
| 30'        | 177        | 587        | 10            | 823           | 413           | 90         | 30'        |
| 40'        | 205        | 620        | 16            | 795           | 380           | 84         | 20'        |
| 50'        | 232        | 654        | 21            | 768           | 346           | 79         | 10'        |
| <b>25°</b> | 9.6 259    | 9.6 687    | 0.04 27       | 0.3 741       | 0.3 313       | 9.95 73    | <b>65°</b> |
| 10'        | 286        | 720        | 33            | 714           | 280           | 67         | 50'        |
| 20'        | 313        | 752        | 39            | 687           | 248           | 61         | 40'        |
| 30'        | 340        | 785        | 45            | 660           | 215           | 55         | 30'        |
| 40'        | 366        | 817        | 51            | 634           | 183           | 49         | 20'        |
| 50'        | 392        | 850        | 57            | 608           | 150           | 43         | 10'        |
| <b>26°</b> | 9.6 418    | 9.6 882    | 0.04 63       | 0.3 582       | 0.3 118       | 9.95 37    | <b>64°</b> |
| 10'        | 444        | 914        | 70            | 556           | 086           | 30         | 50'        |
| 20'        | 470        | 946        | 76            | 530           | 054           | 24         | 40'        |
| 30'        | 495        | 977        | 82            | 505           | 023           | 18         | 30'        |
| 40'        | 521        | 9.7 009    | 88            | 479           | 0.2 991       | 12         | 20'        |
| 50'        | 546        | 040        | 95            | 454           | 960           | 05         | 10'        |
| <b>27°</b> | 9.6 570    | 9.7 072    | 0.05 01       | 0.3 430       | 0.2 928       | 9.94 99    | <b>63°</b> |
| 10'        | 595        | 103        | 08            | 405           | 897           | 92         | 50'        |
| 20'        | 620        | 134        | 14            | 380           | 866           | 86         | 40'        |
| 30'        | 644        | 165        | 21            | 356           | 835           | 79         | 30'        |
| 40'        | 668        | 196        | 27            | 332           | 804           | 73         | 20'        |
| 50'        | 692        | 226        | 34            | 308           | 774           | 66         | 10'        |
| <b>28°</b> | 9.6 716    | 9.7 257    | 0.05 41       | 0.3 284       | 0.2 743       | 9.94 59    | <b>62°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>28°</b> | 9.6 716    | 9.7 257    | 0.05 41       | 0.3 284       | 0.2 743       | 9.94 59    | <b>62°</b> |
| 10'        | 740        | 287        | 47            | 260           | 713           | 53         | 50'        |
| 20'        | 763        | 317        | 54            | 237           | 683           | 46         | 40'        |
| 30'        | 787        | 348        | 61            | 213           | 652           | 39         | 30'        |
| 40'        | 810        | 378        | 68            | 190           | 622           | 32         | 20'        |
| 50'        | 833        | 408        | 75            | 167           | 592           | 25         | 10'        |
| <b>29°</b> | 9.6 856    | 9.7 438    | 0.05 82       | 0.3 144       | 0.2 562       | 9.94 18    | <b>61°</b> |
| 10'        | 878        | 467        | 89            | 122           | 533           | 11         | 50'        |
| 20'        | 901        | 497        | 96            | 099           | 503           | 04         | 40'        |
| 30'        | 923        | 526        | 0.06 03       | 077           | 474           | 9.93 97    | 30'        |
| 40'        | 946        | 556        | 10            | 054           | 444           | 90         | 20'        |
| 50'        | 968        | 585        | 17            | 032           | 415           | 83         | 10'        |
| <b>30°</b> | 9.6 990    | 9.7 614    | 0.06 25       | 0.3 010       | 0.2 386       | 9.93 75    | <b>60°</b> |
| 10'        | 9.7 012    | 644        | 32            | 0.2 988       | 356           | 68         | 50'        |
| 20'        | 033        | 673        | 39            | 967           | 327           | 61         | 40'        |
| 30'        | 055        | 701        | 47            | 945           | 399           | 53         | 30'        |
| 40'        | 076        | 730        | 54            | 924           | 270           | 46         | 20'        |
| 50'        | 097        | 759        | 62            | 903           | 241           | 38         | 10'        |
| <b>31°</b> | 9.7 118    | 9.7 788    | 0.06 69       | 0.2 882       | 0.2 212       | 9.93 31    | <b>59°</b> |
| 10'        | 139        | 816        | 77            | 861           | 184           | 23         | 50'        |
| 20'        | 160        | 845        | 85            | 840           | 155           | 15         | 40'        |
| 30'        | 181        | 873        | 92            | 819           | 127           | 08         | 30'        |
| 40'        | 201        | 902        | 0.07 00       | 799           | 098           | 00         | 20'        |
| 50'        | 222        | 930        | 08            | 778           | 070           | 9.92 92    | 10'        |
| <b>32°</b> | 9.7 242    | 9.7 958    | 0.07 16       | 0.2 758       | 0.2 042       | 9.92 84    | <b>58°</b> |
| 10'        | 262        | 986        | 24            | 738           | 014           | 76         | 50'        |
| 20'        | 282        | 9.8 014    | 32            | 718           | 0.1 986       | 68         | 40'        |
| 30'        | 302        | 042        | 40            | 698           | 958           | 60         | 30'        |
| 40'        | 322        | 070        | 48            | 678           | 930           | 52         | 20'        |
| 50'        | 342        | 097        | 56            | 658           | 903           | 44         | 10'        |
| <b>33°</b> | 9.7 361    | 9.8 125    | 0.07 64       | 0.2 639       | 0.1 875       | 9.92 36    | <b>57°</b> |
| 10'        | 380        | 153        | 72            | 620           | 847           | 28         | 50'        |
| 20'        | 400        | 180        | 81            | 600           | 820           | 19         | 40'        |
| 30'        | 419        | 208        | 89            | 581           | 792           | 11         | 30'        |
| 40'        | 438        | 235        | 97            | 562           | 765           | 03         | 20'        |
| 50'        | 457        | 263        | 0.08 06       | 543           | 737           | 9.91 94    | 10'        |
| <b>34°</b> | 9.7 476    | 9.8 290    | 0.08 14       | 0.2 524       | 0.1 710       | 9.91 86    | <b>56°</b> |
| 10'        | 494        | 317        | 23            | 506           | 683           | 77         | 50'        |
| 20'        | 513        | 344        | 31            | 487           | 656           | 69         | 40'        |
| 30'        | 531        | 371        | 40            | 469           | 629           | 60         | 30'        |
| 40'        | 550        | 398        | 49            | 450           | 602           | 51         | 20'        |
| 50'        | 568        | 425        | 58            | 432           | 575           | 42         | 10'        |
| <b>35°</b> | 9.7 586    | 9.8 452    | 0.08 66       | 0.2 414       | 0.1 548       | 9.91 34    | <b>55°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>35°</b> | 9.7 586    | 9.8 452    | 0.08 66       | 0.2 414       | 0.1 548       | 9.91 34    | <b>55°</b> |
| 10'        | 604        | 479        | 75            | 396           | 521           | 25         | 50'        |
| 20'        | 622        | 506        | 84            | 378           | 494           | 16         | 40'        |
| 30'        | 640        | 533        | 93            | 360           | 467           | 07         | 30'        |
| 40'        | 657        | 559        | 0.09 02       | 343           | 441           | 9.90 98    | 20'        |
| 50'        | 675        | 586        | 11            | 325           | 414           | 89         | 10'        |
| <b>36°</b> | 9.7 692    | 9.8 613    | 0.09 20       | 0.2 308       | 0.1 387       | 9.90 80    | <b>54°</b> |
| 10'        | 710        | 639        | 30            | 290           | 361           | 70         | 50'        |
| 20'        | 727        | 666        | 39            | 273           | 334           | 61         | 40'        |
| 30'        | 744        | 692        | 48            | 256           | 308           | 52         | 30'        |
| 40'        | 761        | 718        | 58            | 239           | 282           | 42         | 20'        |
| 50'        | 778        | 745        | 67            | 222           | 255           | 33         | 10'        |
| <b>37°</b> | 9.7 795    | 9.8 771    | 0.09 77       | 0.2 205       | 0.1 229       | 9.90 23    | <b>53°</b> |
| 10'        | 811        | 797        | 86            | 189           | 203           | 14         | 50'        |
| 20'        | 828        | 824        | 96            | 172           | 176           | 04         | 40'        |
| 30'        | 844        | 850        | 0.10 05       | 156           | 150           | 9.89 95    | 30'        |
| 40'        | 861        | 876        | 15            | 139           | 124           | 85         | 20'        |
| 50'        | 877        | 902        | 25            | 123           | 098           | 75         | 10'        |
| <b>38°</b> | 9.7 893    | 9.8 928    | 0.10 35       | 0.2 107       | 0.1 072       | 9.89 65    | <b>52°</b> |
| 10'        | 910        | 954        | 45            | 090           | 046           | 55         | 50'        |
| 20'        | 926        | 980        | 55            | 074           | 020           | 45         | 40'        |
| 30'        | 941        | 9.9 006    | 65            | 059           | 0.0 994       | 35         | 30'        |
| 40'        | 957        | 032        | 75            | 043           | 968           | 25         | 20'        |
| 50'        | 973        | 058        | 85            | 027           | 942           | 15         | 10'        |
| <b>39°</b> | 9.7 989    | 9.9 084    | 0.10 95       | 0.2 011       | 0.0 916       | 9.89 05    | <b>51°</b> |
| 10'        | 9.8 004    | 110        | 0.11 05       | 0.1 996       | 890           | 9.88 95    | 50'        |
| 20'        | 020        | 135        | 16            | 980           | 865           | 84         | 40'        |
| 30'        | 035        | 161        | 26            | 965           | 839           | 74         | 30'        |
| 40'        | 050        | 187        | 36            | 950           | 813           | 64         | 20'        |
| 50'        | 066        | 212        | 47            | 934           | 788           | 53         | 10'        |
| <b>40°</b> | 9.8 081    | 9.9 238    | 0.11 57       | 0.1 919       | 0.0 762       | 9.88 43    | <b>50°</b> |
| 10'        | 096        | 264        | 68            | 904           | 736           | 32         | 50'        |
| 20'        | 111        | 289        | 79            | 889           | 711           | 21         | 40'        |
| 30'        | 125        | 315        | 90            | 875           | 685           | 10         | 30'        |
| 40'        | 140        | 341        | 0.12 00       | 860           | 659           | 00         | 20'        |
| 50'        | 155        | 366        | 11            | 845           | 634           | 9.87 89    | 10'        |
| <b>41°</b> | 9.8 169    | 9.9 392    | 0.12 22       | 0.1 831       | 0.0 608       | 9.87 78    | <b>49°</b> |
| 10'        | 184        | 417        | 33            | 816           | 583           | 67         | 50'        |
| 20'        | 198        | 443        | 44            | 802           | 557           | 56         | 40'        |
| 30'        | 213        | 468        | 55            | 787           | 532           | 45         | 30'        |
| 40'        | 227        | 494        | 67            | 773           | 506           | 33         | 20'        |
| 50'        | 241        | 519        | 78            | 759           | 481           | 22         | 10'        |
| <b>42°</b> | 9.8 255    | 9.9 544    | 0.12 89       | 0.1 745       | 0.0 456       | 9.87 11    | <b>48°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

| $x$        | $L \sin x$ | $L \tan x$ | $\log \sec x$ | $\log \csc x$ | $\log \cot x$ | $L \cos x$ |            |
|------------|------------|------------|---------------|---------------|---------------|------------|------------|
| <b>42°</b> | 9.8 255    | 9.9 544    | 0.12 89       | 0.1 745       | 0.0 456       | 9.87 11    | <b>48°</b> |
| 10'        | 269        | 570        | 0.13 01       | 731           | 430           | 9.86 99    | 50'        |
| 20'        | 283        | 595        | 12            | 717           | 405           | 88         | 40'        |
| 30'        | 297        | 621        | 24            | 703           | 379           | 76         | 30'        |
| 40'        | 311        | 646        | 35            | 689           | 354           | 65         | 20'        |
| 50'        | 324        | 671        | 47            | 676           | 329           | 53         | 10'        |
| <b>43°</b> | 9.8 338    | 9.9 697    | 0.13 59       | 0.1 662       | 0.0 303       | 9.86 41    | <b>47°</b> |
| 10'        | 351        | 722        | 71            | 649           | 278           | 29         | 50'        |
| 20'        | 365        | 747        | 82            | 635           | 253           | 18         | 40'        |
| 30'        | 378        | 772        | 94            | 622           | 228           | 06         | 30'        |
| 40'        | 391        | 798        | 0.14 06       | 609           | 202           | 9.85 94    | 20'        |
| 50'        | 405        | 823        | 18            | 595           | 177           | 82         | 10'        |
| <b>44°</b> | 9.8 418    | 9.9 848    | 0.14 31       | 0.1 582       | 0.0 152       | 9.85 69    | <b>46°</b> |
| 10'        | 431        | 874        | 43            | 569           | 126           | 57         | 50'        |
| 20'        | 444        | 899        | 55            | 556           | 101           | 45         | 40'        |
| 30'        | 457        | 9.9 924    | 68            | 543           | 0.0 076       | 32         | 30'        |
| 40'        | 469        | 949        | 80            | 531           | 051           | 20         | 20'        |
| 50'        | 482        | 975        | 93            | 518           | 025           | 07         | 10'        |
| <b>45°</b> | 9.8 495    | 10.0 000   | 0.15 05       | 0.1 505       | 0.0 000       | 9.84 95    | <b>45°</b> |
|            | $L \cos y$ | $L \cot y$ | $\log \csc y$ | $\log \sec y$ | $\log \tan y$ | $L \sin y$ | $y$        |

## FORMULÆ.

$$a \csc A = b \csc B = c \csc C.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$\cot B = c \frac{\csc A}{b} - \cot A.$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\cot \frac{1}{2} A = \frac{s-a}{r}; \quad \cot \frac{1}{2} B = \frac{s-b}{r}.$$

$$\text{Circle: circum} = 2\pi r; \text{ area} = \pi r^2.$$

$$\text{sector} = \frac{1}{2} r \times \text{arc}; \text{ seg} = \frac{1}{2} r^2 (\theta - \sin \theta).$$

$$\text{Sphere: sur} = 4\pi R^2; \text{ vol} = \frac{4}{3} \pi R^3.$$

$$\text{zone} = 2\pi Rh; \text{ seg} = \frac{1}{3} \pi h^2 (3R - h).$$

$$\text{Natural log } N = 2.30259 \log_{10} N.$$

$$\text{Pendulum: } \pi^2 l = gt.$$

$$\text{Falling bodies: } 2s = gt^2 = vt = v^2 \div g.$$

## CONSTANTS.

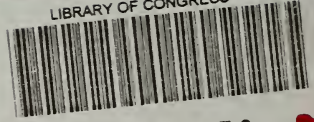
|               | NUMBER.       | LOG.    |
|---------------|---------------|---------|
| radian        | 57.29578°     | 1.75812 |
| $\pi$         | 180°          | 2.25527 |
| $\pi$         | 10800'        | 4.03342 |
| $\pi$         | 648000''      | 5.81158 |
| $\pi$         | 3.14159       | 0.49715 |
| $1 \div \pi$  | 0.31831       | 9.50285 |
| $\pi^2$       | 9.86960       | 0.99430 |
| $\sqrt{\pi}$  | 1.77245       | 0.24857 |
| 1 mile        | 5280 ft.      | 3.72263 |
| 1 acre        | 43560 sq. ft. | 4.63909 |
| $e$           | 2.71828       | 0.43429 |
| $1 \div M$    | 2.30259       | 0.36221 |
| vel. light    | 186330        | 5.27028 |
| $l(40^\circ)$ | 39.0986       | 1.59216 |
| $g(40^\circ)$ | 32.1573       | 1.50728 |







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